MODUL_2022

Machine Component Design and Mechanics

Dávid CSONKA // Gyula VASVÁRI

Design and calculation of machine components

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INTRODUCTION

Understanding the foundational concepts of statics and dynamics is crucial for engineers and physicists working on various mechanical systems and structures. In the vast realm of mechanical engineering and physics, the principles of statics and dynamics stand as cornerstone disciplines. They offer invaluable insights into understanding, predicting, and controlling the behaviors of objects, from stationary to those in complex motion. From the fundamental force systems that dictate equilibrium states to the intricate dance of bodies in motion, these principles form the bedrock upon which many modern engineering marvels are built. Whether you're delving into the effects of loads on beams, the nuances of friction, or the intricate ballet of kinematics and kinetics, a foundational grasp of these areas is imperative. Additionally, understanding collisions, both central and otherwise, and their resulting interactions becomes vital in many real-world applications. This comprehensive overview aims to shed light on these essential topics, paving the way for deeper exploration and mastery.

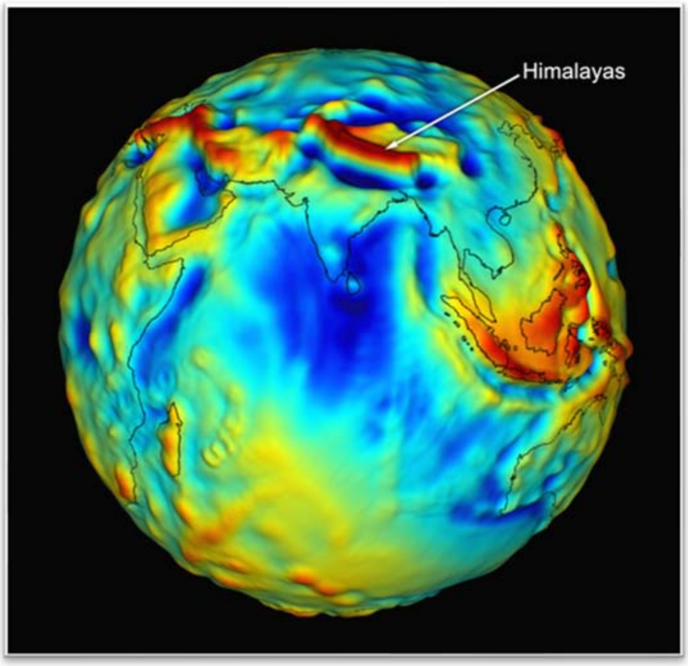
Basic knowledge necessary for performing calculations in the field of statics and dynamics. Types of force systems, calculating resultant forces, equilibrium. Loads, load diagrams in beams. Cases of friction.

Kinematics of center of gravity and rigid body. Kinematic characteristics of motion. Examination of special movements. Elemental movements of a rigid body. Finite motion of a rigid body.

Kine-matics of structures. Kinetics of material point and rigid body. Foundation of kinetic. Free and forced movements. Moments of inertia. Rigid body pulse, pulse moment, kinetic energy. Impulse theorem, vortex theorem. Energy and work. Collision of bodies. Central collision, collision dia-gram.

1 CENTER OF GRAVITY – CENTROID

Every object on Earth is affected by gravity. Because this effect acts on every part of the body, it becomes a spatially distributed force system. This force system can be treated parallel, as the distance from the centre of the Earth is large compared to the size of the bodies under study.



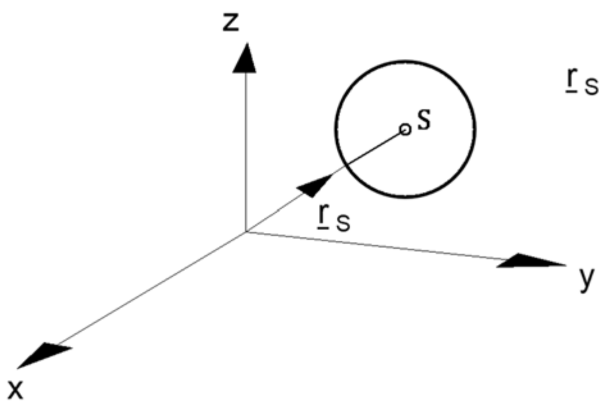
The gravity strength distribution of Earth

This force system can be substituted with its resultant. The acting point of this resultant force is the centre of mass of the object. The centre of mass of a geometric object of uniform density is called centroid.

The waiter's hand is under the center of mass of the system

1.1 CENTROID LOCATION CALCULATION

The location vector can be calculated as follows:



$$\underline{r}_S = \frac{\int \underline{r}\ dm}{\int dm} = \frac{\int \underline{r}\ dm}{m}$$

where:

- r_S is the location vector of the centroid;
- m is the mass of the body;
- r is the location vector of the mass element dm.

In case of uniform density:

r = constant

m = r *V

dm = r *dV

$$\underline{r}_S = \frac{\int \underline{r}\ dm}{\int dm} = \frac{\int \underline{r}\ \rho\ dV}{\int \rho\ dV} = \frac{\int \underline{r}\ dV}{\int dV} = \frac{\sum \underline{r}_i V_i}{\sum V_i}$$

Thus:

$$x_S = \frac{\int x\ dV}{\int dV} = \frac{\sum x_i V_i}{\sum V_i}$$

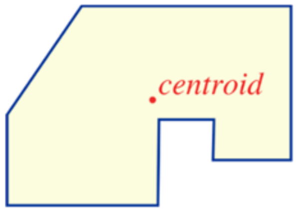
$$y_S = \frac{\int y\ dV}{\int dV} = \frac{\sum y_i V_i}{\sum V_i}$$

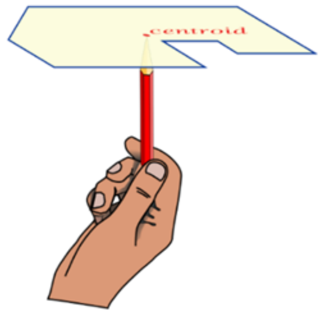
$$z_S = \frac{\int z\ dV}{\int dV} = \frac{\sum z_i V_i}{\sum V_i}$$

Each coordinate of the location vector of the centroid can be given as the summation of the product of the volume and coordinate of each part divided by the total volume.

1.2 CENTROID OF PLANAR SHAPES

Significant dimension is surface area.





The calculation of the centroid of planar shapes is produced by the area like so:

[redacted]

Thus:

$$x_S = \frac{\int x \, dA}{\int dA} = \frac{\sum x_i A_i}{\sum A_i}$$

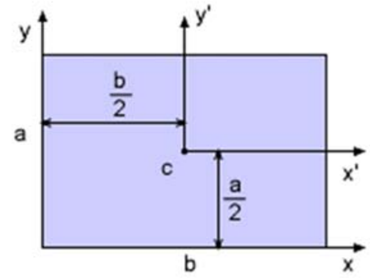
$$y_S = \frac{\int y \, dA}{\int dA} = \frac{\sum y_i A_i}{\sum A_i}$$

$$z_S = \frac{\int z \, dA}{\int dA} = \frac{\sum z_i A_i}{\sum A_i}$$

Each coordinate of the location vector of the centroid can be given as the summation of the product of the area and coordinate of each part divided by the total area.

But in order to do so, the centroid coordinates of certain basic planar shapes should be known.

1.2.1 CENTROID OF BASIC PLANAR SHAPES

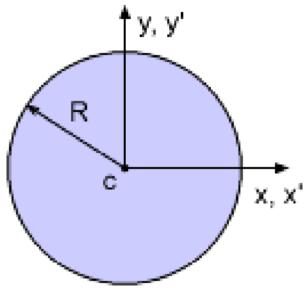


Rectangle:

$A = a * b;$

$x_s = \frac{b}{2}$

$y_s = \frac{h}{2};$

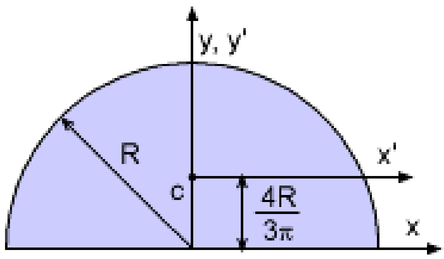


Circle:

$A = r^2\pi;$

$x_s = 0;$

$y_s = 0;$

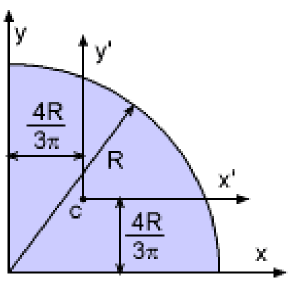


Semicircle:

$A = \frac{r^2\pi}{2};$

$x_s = 0;$

$y_s = \frac{4r}{3\pi};$

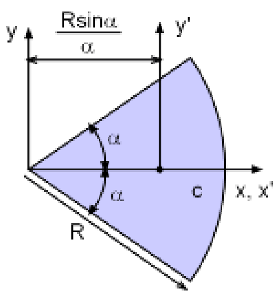


Quarter circle:

$A = \frac{r^2\pi}{4};$

$x_s = \frac{4r}{3\pi};$

$y_s = \frac{4r}{3\pi};$



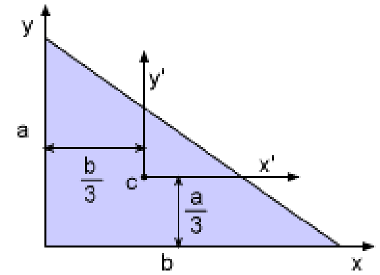
Segment of a circle:

$A = \alpha * r^2;$

$x_s = \frac{R * sin\alpha}{\alpha};$

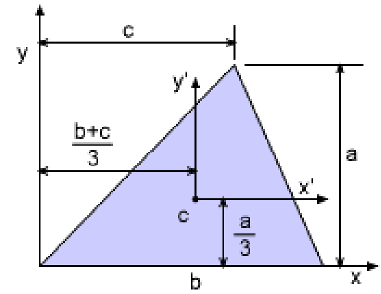
$y_s = 0;$

All angles are in radians!



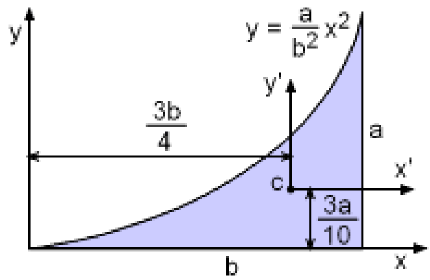
Right triangle:

$$A = \frac{b*h}{2};$$
$$x_s = \frac{b}{3};$$
$$y_s = \frac{h}{3};$$



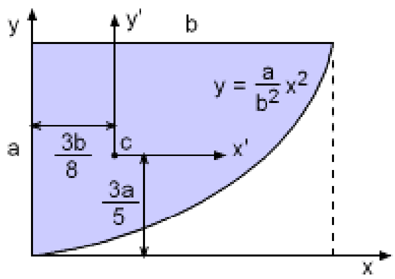
Triangle:

$$A = \frac{a*b}{2};$$
$$x_s = \frac{b+c}{3};$$
$$y_s = \frac{a}{3};$$



Parabolic Spandrel:

$$A = \frac{a*b}{3};$$
$$x_s = \frac{3*b}{4};$$
$$y_s = \frac{3*a}{10};$$

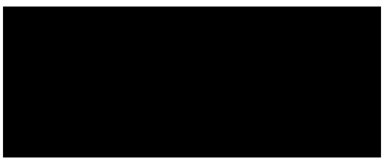


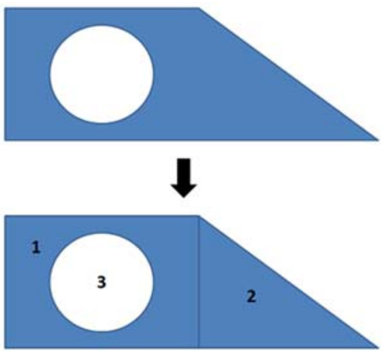
Semi Parabolic:

$$A = \frac{2*a*b}{3};$$
$$x_s = \frac{3*b}{8};$$
$$y_s = \frac{3*a}{5};$$

1.2.2 CENTROID OF COMPOSITE PLANAR SHAPES

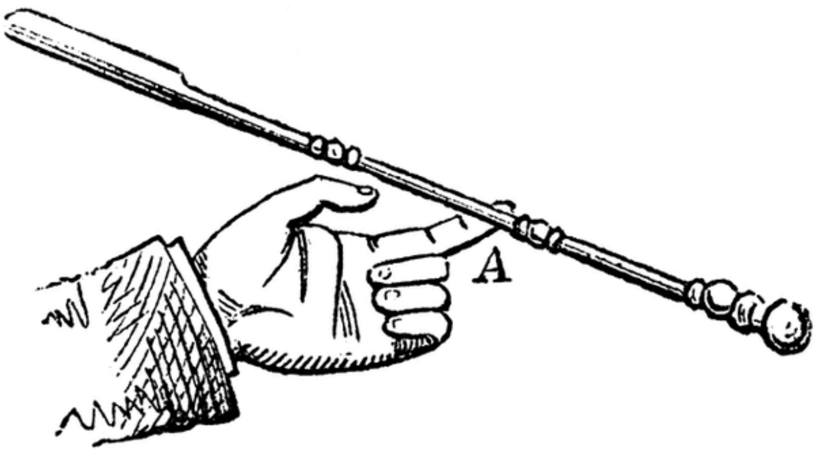
As discussed earlier, the centroid of a composite planar shape can be calculated by dividing the shape into basic parts.





1.3 CENTROID OF LINES

Significant dimension: length



The calculation of the centroid of lines is produced by the length like so:

[redacted]

Thus:

$$x_S = \frac{\int x \, dl}{\int dl} = \frac{\sum x_i l_i}{\sum l_i}$$

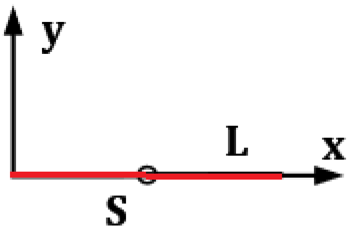
$$y_S = \frac{\int y \, dl}{\int dl} = \frac{\sum y_i l_i}{\sum l_i}$$

$$z_S = \frac{\int z \, dl}{\int dl} = \frac{\sum z_i l_i}{\sum l_i}$$

Each coordinate of the location vector of the centroid can be given as the summation of the product of the length and coordinate of each part divided by the total length.

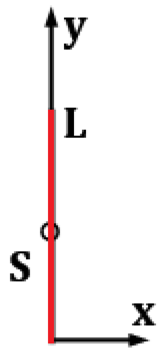
But in order to do so, the centroid coordinates of certain basic lines should be known.

1.3.1 CENTROID OF BASIC LINES



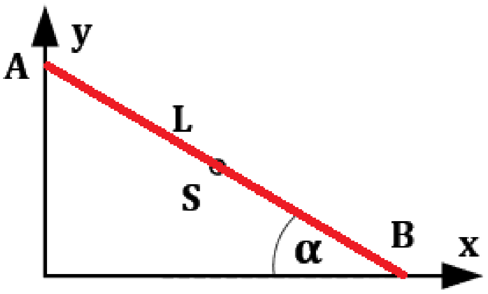
Horizontal line:

$x_s = L/2;$
$y_s = 0$



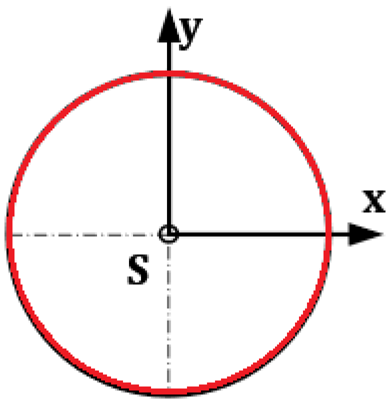
Vertical line:

$x_s = 0;$
$y_s = L/2$



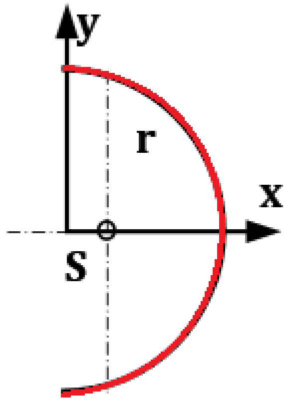
General line:

$$x_s = \frac{L \cdot \cos\alpha}{2}; \text{ or } x_s = \frac{x_A + x_B}{2};$$
$$y_s = \frac{L \cdot \sin\alpha}{2}; \text{ or } y_s = \frac{y_A + y_B}{2}$$



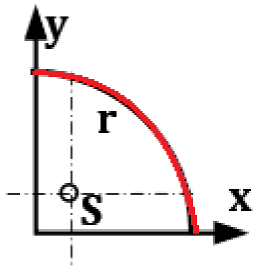
Circle:

$x_s = 0;$
$y_s = 0$



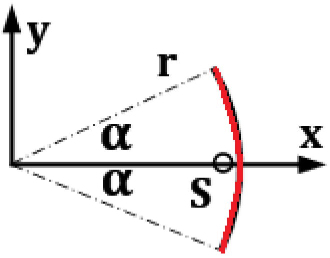
Semicircle arc:

$$x_S = \frac{2r}{\pi};$$
$$y_S = 0;$$



Quarter circle arc:

$$x_S = \frac{2r}{\pi};$$
$$y_S = \frac{2r}{\pi};$$



General arc:

$$x_S = \frac{r \cdot \sin\alpha}{\alpha};$$
$$y_S = 0;$$

1.3.2 CENTROID OF COMPOSITE LINES

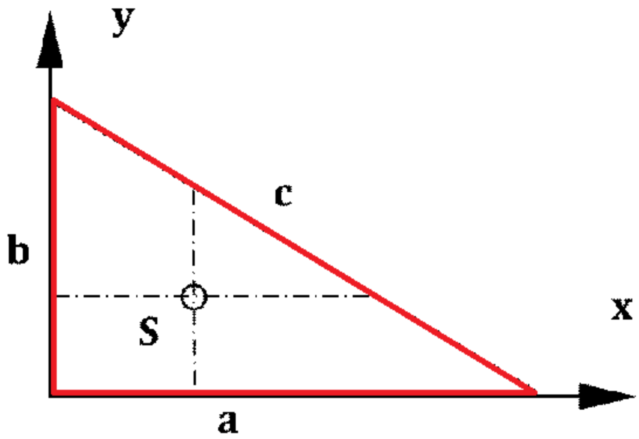
As discussed earlier, the centroid of a composite line can be calculated by dividing the line basic segments.

[redacted]

Example:

$$x_s = \frac{x_a * a + x_b * b + x_c * c}{a + b + c}$$

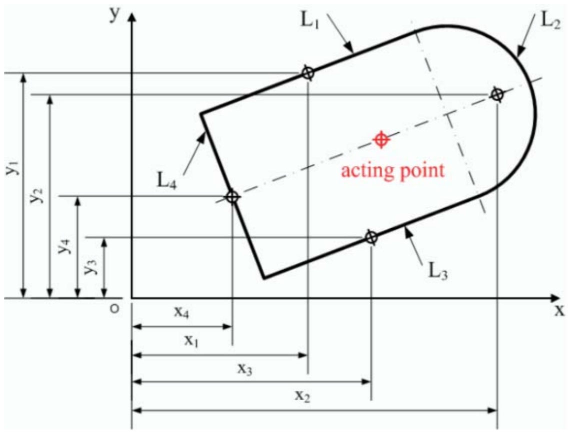
$$y_s = \frac{y_a * a + y_b * b + y_c * c}{a + b + c}$$



1.3.3 EXAMPLES FOR THE PRACTICAL USE OF LINE CENTROID CALCULATIONS:

CUTTER

The most common example is calculating the acting point of a cutting process of a stamp cutter. Thus, the hydraulic piston can be precisely positioned avoiding any bending load.



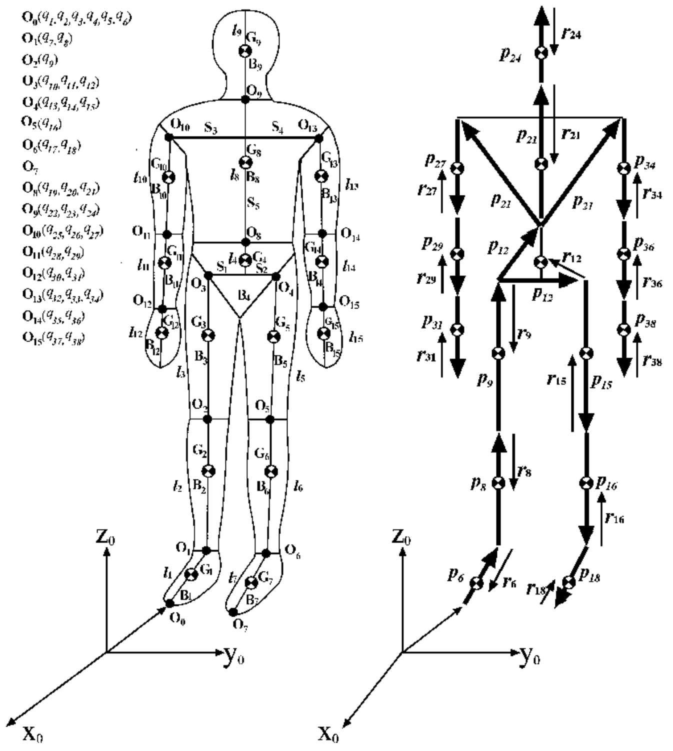
TRUSSES

Any prismatic component can be treated as a line when calculating center of mass. It simplifies the calculation a lot.

SUBSTITUTION

Simplifying the centroid calculation of complex structures can be achieved by substituting the segments with weighted lines and using the superpositon principle.

Even the human body can be substituted with segments separated at the joints, thus being possible to calcualte the center of mass in any given body position.

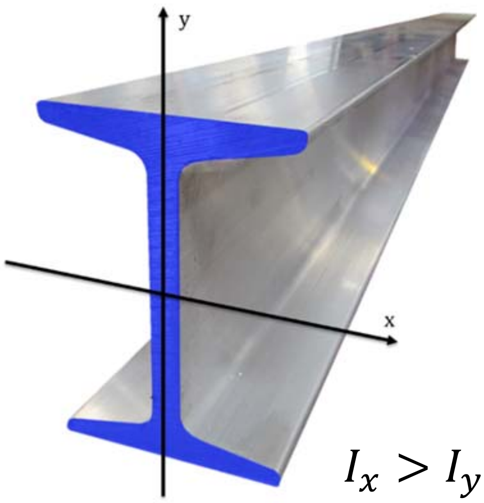


2 SECOND MOMENT OF AREA

2.1 WHAT IS IT GOOD FOR?

1. Bending resistance of cross sections.
2. Selection of the appropriate beam.
3. Determining the best installation position.
4. Strength analysis.

Mechanical design is impossible without it.



2.2 DEFINITION

„Zero order” moment of area (area):

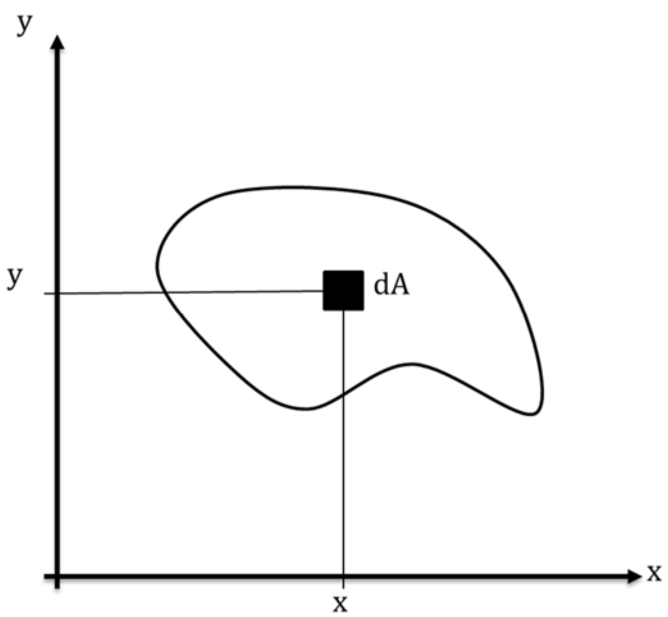
$$A = \int_{(A)} x^0 dA$$

First order (static) moment of area (moment):

$$S_y = \int_{(A)} x dA$$

Second order moment of area (area moment of inertia):

$$I_y = \int_{(A)} x^2 dA$$



2.3 SECOND MOMENT OF AREA ON AN AXIS:

(equatorial moment of area)

The second moment of area on an axis of a planar object is the surface integral of the given area, in which the surface elements are multiplied by the square of their distance from the axis. The index of the letter I indicates the axis to which the second moment of area applies.

$$I_y = \int_{(A)} x^2 dA$$

and

$$I_x = \int_{(A)} y^2 dA$$

The result is scalar, its value can only be positive.

2.4 SECOND MOMENT OF AREA ON TWO PERPENDICULAR AXES:

(centrifugal second moment of area)

The second moment of area on a pair of perpendicular axes of a planar object is the surface integral of the given area, in which the surface elements are multiplied by their distances from the axes. The two indices of the letter I indicate the pair of axles to which the second moment of area applies.

$$I_{xy} = \int_{(A)} xydA$$

The result is scalar, its value can be positive or negative.

2.5 SECOND MOMENT OF AREA TENSOR:

$$[I_s] = \begin{bmatrix} I_x & -I_{xy} \\ -I_{yx} & I_y \end{bmatrix}$$

The standard dimensions of second moment of area: m^4, cm^4, mm^4.

2.6 SECOND MOMENT OF AREA ON A POINT:

(polar second moment of area)

The second moment of area of a planar object on a point is a surface integral of the given area, in which the surface elements are multiplied by the square of their distance from the point. The index of the letter I indicates the point to which the second moment of area applies.

$$I_O = \int_{(A)} r^2 dA = I_x + I_y$$

The result is scalar, its value can only be positive.

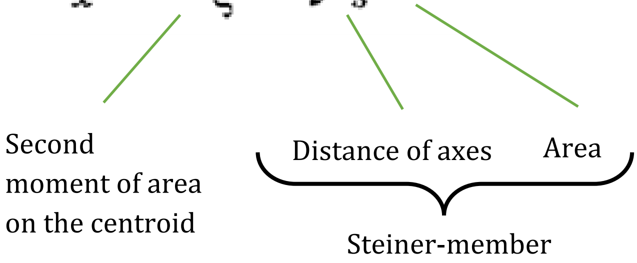
$$I_0 = \int_{(A)} r^2 \mathrm{d}A = \int_{(A)} (x^2 + y^2)\mathrm{d}A = \int_{(A)} x^2 \mathrm{d}A + \int_{(A)} y^2 \mathrm{d}A = I_y + I_x$$

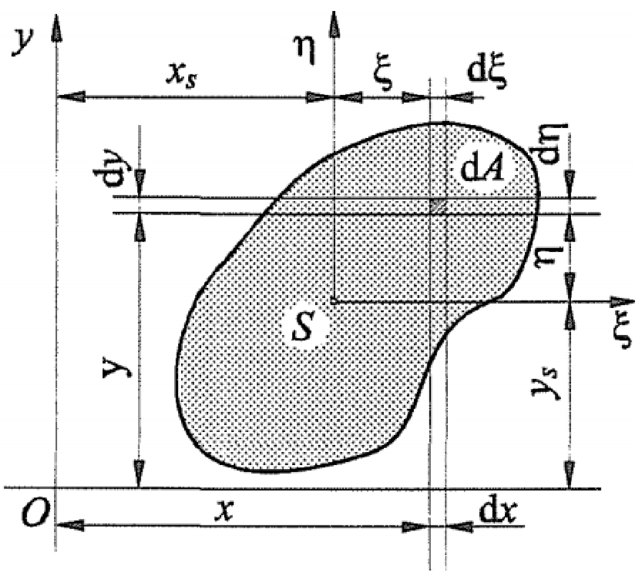
2.7 PARALLEL COORDINATE TRANSFORMATION

2.7.1 STEINER'S THEOREM FOR EQUATORIAL AXES:

The second moment of area of a planar object on any axis is obtained by adding the product of the square of the distance between the axes and the area of the plane to the second moment of area on a parallel axis which is passing through the center of gravity of the plane.

$$I_x = I_\xi + y_s^2 A.$$





Thus,

$$I_y = I_\eta + x_s^2 A.$$

2.7.2 STEINER THEOREM FOR PERPENDICULAR AXIS PAIR:

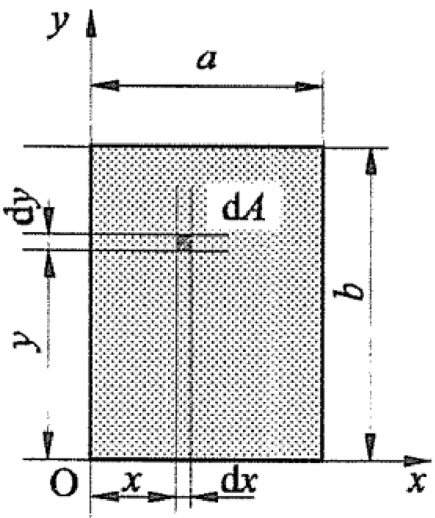
The (centrifugal) second moment of area of a planar object at any perpendicular pair of axes is obtained by adding the product of the area of the object and the distance coordinates of the axes to the centrifugal second moment of area on the centroid.

$$I_{xy} = I_{\xi\eta} + x_s y_s A.$$

Second moment of area on the centroid — $I_{\xi\eta}$

Distances of axes — $x_s y_s$; Area — A; together: Steiner-member

2.8 SECOND MOMENT OF AREA OF A RECTANGLE



On x axis:

$$I_x = \int_{(A)} y^2 \mathrm{d}A = \int_0^a \int_0^b y^2 \mathrm{d}y\,\mathrm{d}x = \int_0^a \mathrm{d}x \int_0^b y^2 \mathrm{d}y = \frac{ab^3}{3}.$$

On y axis:

$$I_y = \frac{ba^3}{3}.$$

On xy pair of axes:

$$I_{xy} = \int_{(A)} xy\,\mathrm{d}A = \int_0^a \int_0^b xy\,\mathrm{d}x\,\mathrm{d}y = \int_0^a x\,\mathrm{d}x \int_0^b y\,\mathrm{d}y = \frac{a^2 b^2}{4}.$$

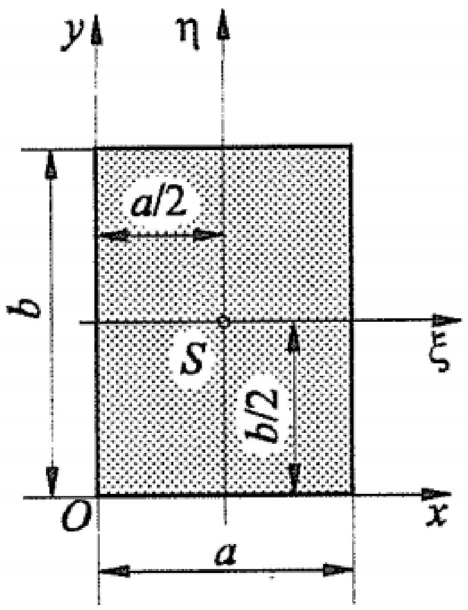
On centroidal axes:

$$I_\xi = \frac{ab^3}{3} - \left(\frac{b}{2}\right)^2 ab = \frac{ab^3}{3} - \frac{ab^3}{4} = \frac{ab^3}{12}.$$

$$I_\eta = I_y - x_s^2 A = \frac{a^3 b}{3} - \left(\frac{a}{2}\right)^2 ab = \frac{a^3 b}{3} - \frac{a^3 b}{4} = \frac{a^3 b}{12}.$$

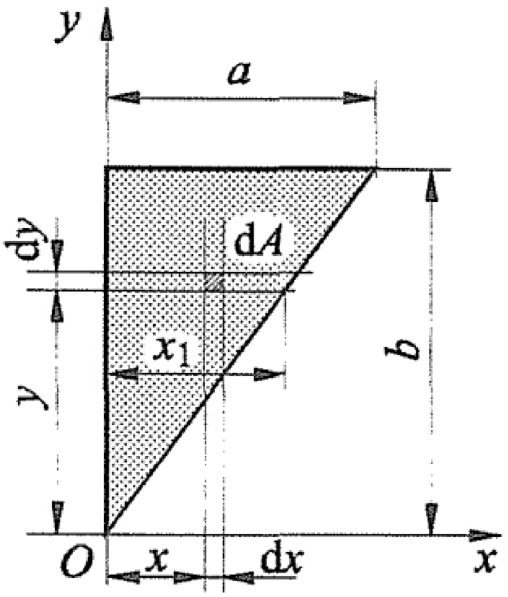
Centrifugal second moment of area:

$$I_{\xi\eta} = I_{xy} - x_s y_s A = \frac{a^2 b^2}{4} - \frac{a}{2}\frac{b}{2} ab = 0 .$$



2.9 SECOND MOMENT OF AREA OF A RIGHT TRIANGLE

On x axis:



$$I_x = \int_{(A)} y^2 \mathrm{d}A = \int_0^b y^2 \left(\int_0^{x_1} \mathrm{d}x \right) \mathrm{d}y =$$

$$\int_0^b y^2 x_1 \mathrm{d}y = \int_0^b y^2 \frac{a}{b} y \mathrm{d}y = \frac{ab^3}{4}.$$

On y axis:

$$I_y = \int_{(A)} x^2 \mathrm{d}A = \int_0^b \left[\int_0^{x_1} x^2 \mathrm{d}x \right] \mathrm{d}y = \int_0^b \frac{x_1^3}{3} \mathrm{d}y = \int_0^b \frac{1}{3}\left(\frac{a}{b} y\right)^3 \mathrm{d}y = \frac{a^3 b}{12}.$$

On xy pair of axes:

$$I_{xy} = \int_{(A)} xy \mathrm{d}A = \int_0^b y \left[\int_0^{x_1} x^2 \mathrm{d}x \right] \mathrm{d}y = \int_0^b y \frac{x_1^2}{2} \mathrm{d}y = \int_0^b y \frac{1}{2}\left(\frac{a}{b} y\right)^2 \mathrm{d}y = \frac{a^2 b^2}{8}$$

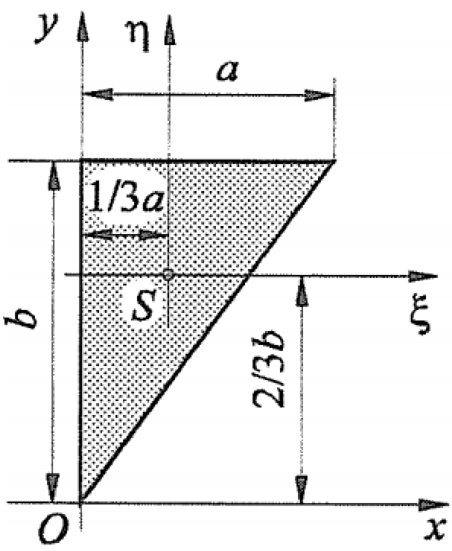
On centroidal axes:

$$I_x = I_\xi + y_s^2 A; \qquad I_\xi = I_x - y_s^2 A = \frac{ab^3}{4} - \left(\frac{2}{3}b\right)^2 \frac{1}{2}ab = \frac{ab^3}{36}.$$

$$I_y = I_\eta + y_s^2 A; \qquad I_\eta = I_y - y_s^2 A = \frac{a^3 b}{12} - \left(\frac{1}{3}a\right)^2 \frac{1}{2}ab = \frac{a^3 b}{36}.$$

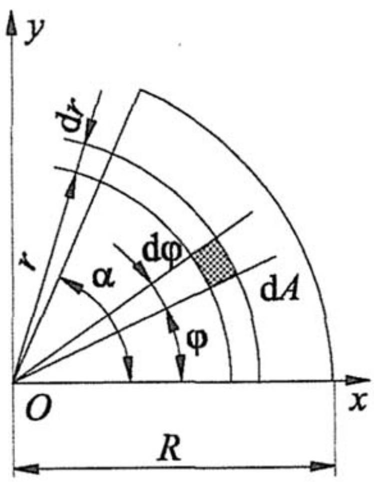
Centrifugal second moment of area:

$$I_{\xi\eta} = I_{xy} - x_s y_s A = \frac{a^2b^2}{8} - \left(\frac{a}{3}\right)\left(\frac{2}{3}b\right)\frac{1}{2}ab = \frac{a^2b^2}{72}.$$



2.10 SECOND MOMENT OF AREA OF A SECTOR

On x axis:



$$I_x = \int_{(A)} y^2 \mathrm{d}A = \iint_{(A)} (r \cdot \sin\varphi)^2 r \mathrm{d}\varphi \mathrm{d}r = \int_0^R r^3 \mathrm{d}r \int_0^\alpha \sin^2\varphi \, \mathrm{d}\varphi = \frac{R^4}{4}\left(\frac{\alpha}{2} - \frac{1}{4}\sin 2\alpha\right),$$

$$I_x = \frac{R^4}{16}(2\alpha - \sin 2\alpha).$$

On y axis:

$$I_y = \int_{(A)} x^2 \mathrm{d}A = \iint_{(A)} (r \cdot \cos\varphi)^2 r \mathrm{d}\varphi \mathrm{d}r = \int_0^R r^3 \mathrm{d}r \int_0^\alpha \cos^2\varphi \mathrm{d}\varphi = \frac{R^4}{4}\left(\frac{\alpha}{2} + \frac{1}{4}\sin 2\alpha\right).$$

On xy pair of axes:

$$I_{xy} = \int_{(A)} xy \mathrm{d}A = \iint_{(A)} r\cos\varphi r \sin\varphi r \mathrm{d}\varphi \mathrm{d}r = \int_0^R r^3 \mathrm{d}r \int_0^a \sin\varphi \cos\varphi \mathrm{d}\varphi =$$

$$= \frac{R^4}{4}\frac{1}{2}\sin^2\alpha = \frac{R^4}{8}\sin^2\alpha.$$

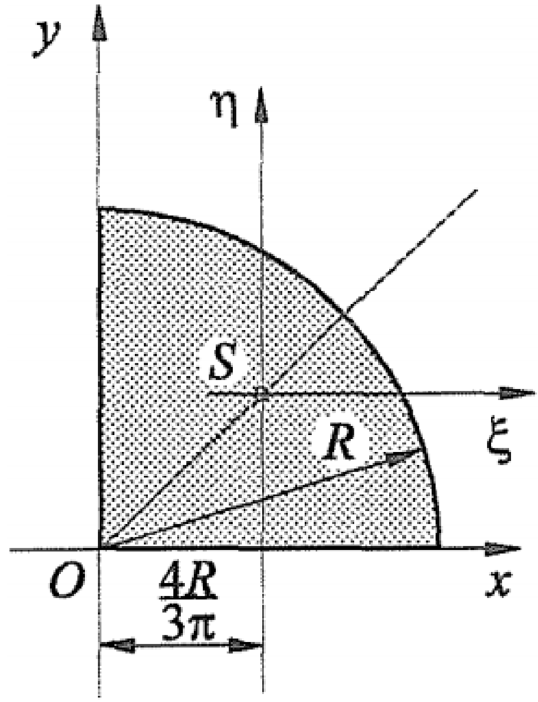
On point O:

$$I_O = \int_{(A)} r^2 \mathrm{d}A = \iint_{(A)} r^2 \cdot r \cdot \mathrm{d}\varphi \cdot \mathrm{d}r = \int_0^R r^3 \mathrm{d}r \int_0^\alpha \mathrm{d}\varphi = \frac{R^4}{4}\alpha.$$

2.11 SECOND MOMENT OF AREA OF A QUARTER CIRCLE

On x and y axis:

$$I_x = \frac{R^4}{16}(2\alpha - \sin 2\alpha) = \frac{R^4\pi}{16}.$$



On centroid axes:

$$I_\xi = I_x - y_s^2 A = \frac{R^4\pi}{16} - \left(\frac{4R}{3\pi}\right)^2 \cdot \frac{R^2\pi}{4} =$$
$$= \frac{R^4\pi}{16}\left(1 - \frac{64}{9\pi^2}\right).$$

Because of symmetry:

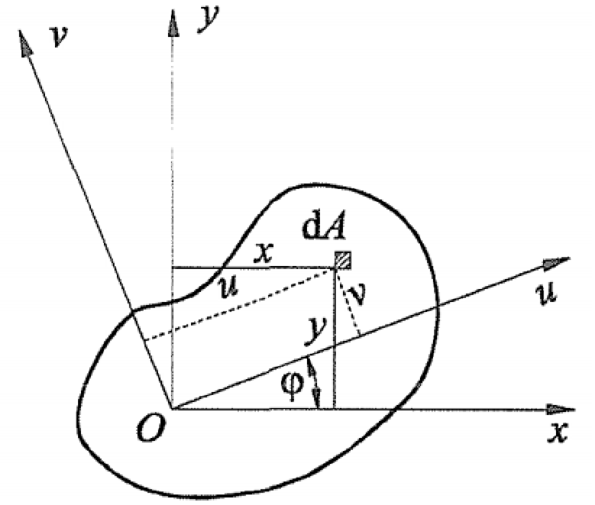
$$I_\xi = I_\eta .$$

Centrifugal second moment of area:

$$I_{xy} = \frac{R^4}{8}\sin^2\alpha = \frac{R^4}{8},$$

$$I_{\xi\eta} = I_{xy} - x_s y_s A = \frac{R^4}{8} - \left(\frac{4R}{3\pi}\right)\left(\frac{4R}{3\pi}\right)\frac{R^2\pi}{4} = \frac{R^4}{8}\left(1 - \frac{32}{9\pi}\right) < 0.$$

2.12 SECOND MOMENT OF AREA ON ROTATED AXES



On rotated u axis:

$$I_u = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2}\cos 2\varphi - I_{xy}\sin 2\varphi$$

On rotated v axis:

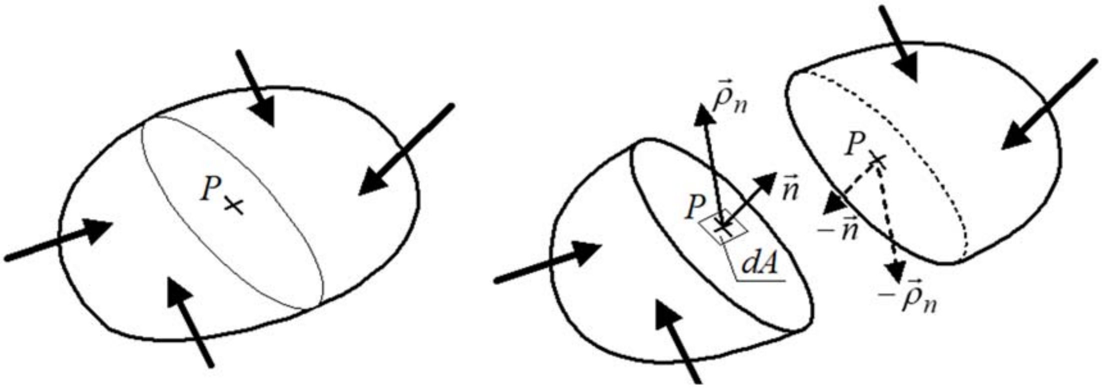
$$I_v = \frac{I_x + I_y}{2} - \frac{I_x - I_y}{2}\cos 2\varphi + I_{yx}\sin 2\varphi$$

On rotated pair of uv axes:

$$I_{uv} = \frac{I_x - I_y}{2}\sin 2\varphi + I_{xy}\cos 2\varphi$$

3 THE CONCEPT AND REPRESENTATION OF STRESS

3.1 STRESS VECTOR



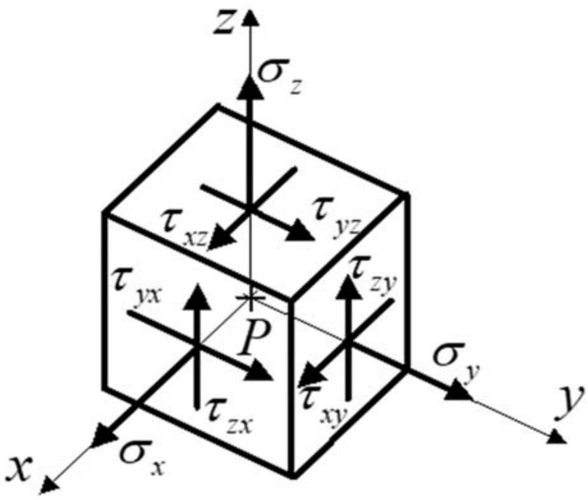
Definition: The intensity vector of the internal forces distributed on a cross-section.

Sign: σ

Dimension: Pa = $\frac{N}{m^2}$ or MPa = $\frac{N}{mm^2}$

3.2 STRESS TENSOR

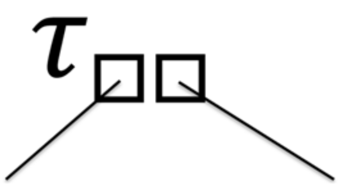
$$\overline{\overline{F_P}} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}$$



Normal stresses: $\sigma_x;\ \sigma_y;\ \sigma_z$

Shear stresses: $\tau_{xy} = \tau_{yx};\ \tau_{yz} = \tau_{zy};$

Indexes of shear stress



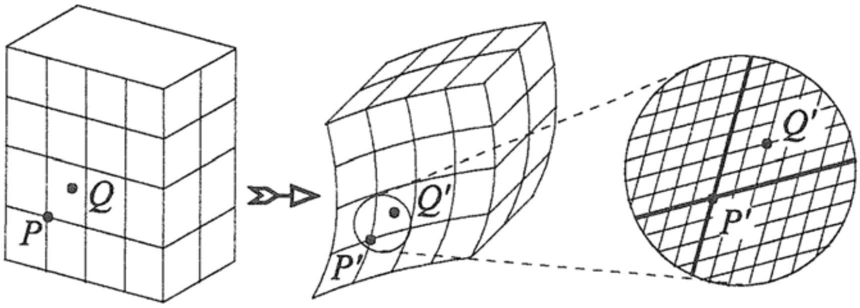
Direction of stress Plane of stress

4 THE CONCEPT AND REPRESENTATION OF DEFORMATION

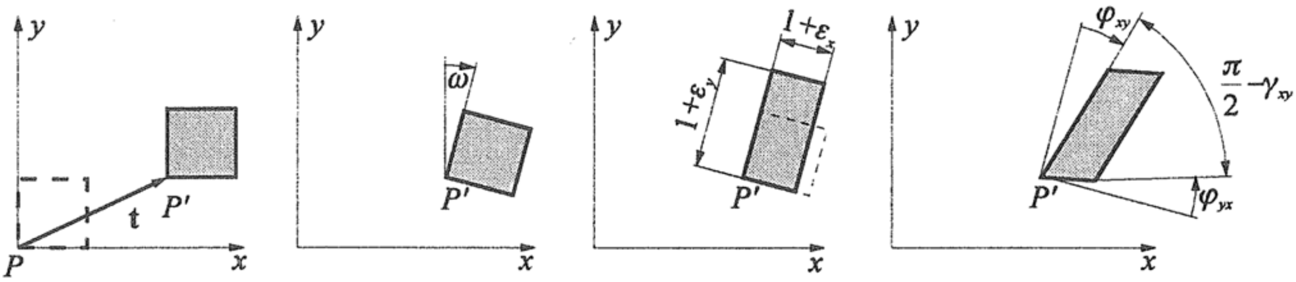
4.1 SPECIFIC ELONGATION

$$\varepsilon = \frac{\Delta l}{l_0}$$

The change relative to the original distance between two points in the material.



Specific angle change: γ : the angle change of two perpendicular axes in the material



Deformation can be translation and rotation.

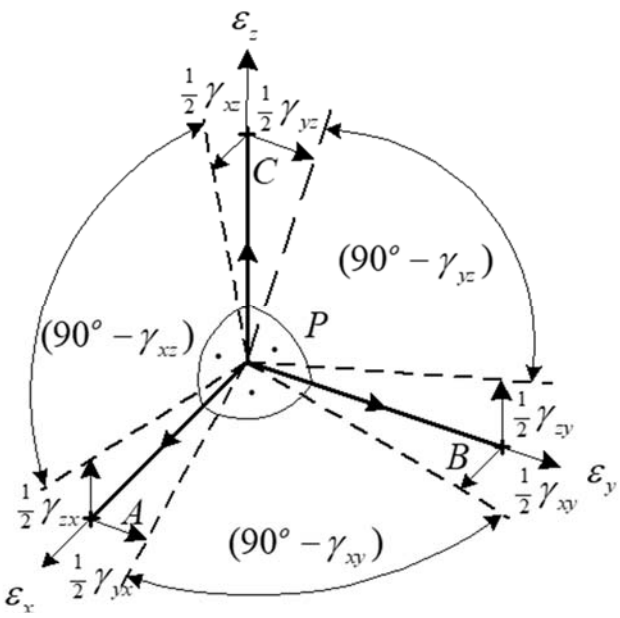
4.2 DEFORMATION TENSOR

$$\left[\underline{\underline{A}}_P\right]=\begin{bmatrix} \varepsilon_x & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{yx} & \varepsilon_y & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{zx} & \frac{1}{2}\gamma_{zy} & \varepsilon_z \end{bmatrix}$$

Specific elongations: ε_x; ε_y; ε_z.

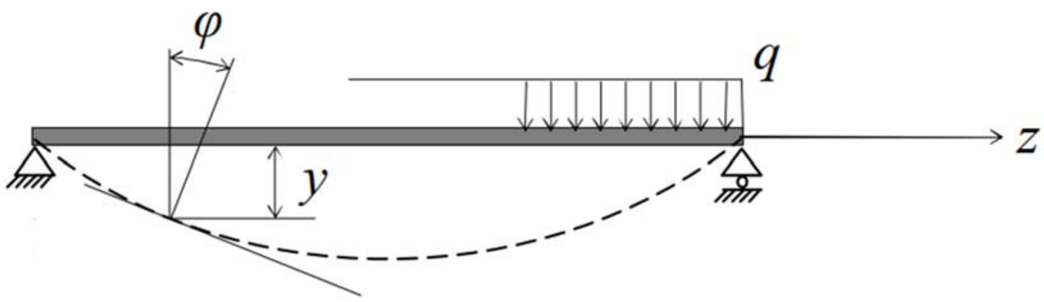
Specific angle changes:

$\gamma_{xy} = \gamma_{yx}$; $\gamma_{yz} = \gamma_{zy}$; $\gamma_{xz} = \gamma_{zx}$;

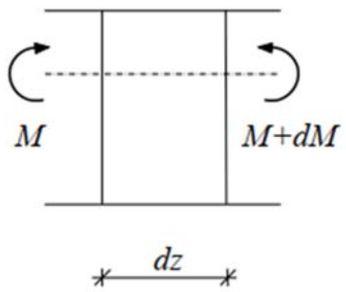


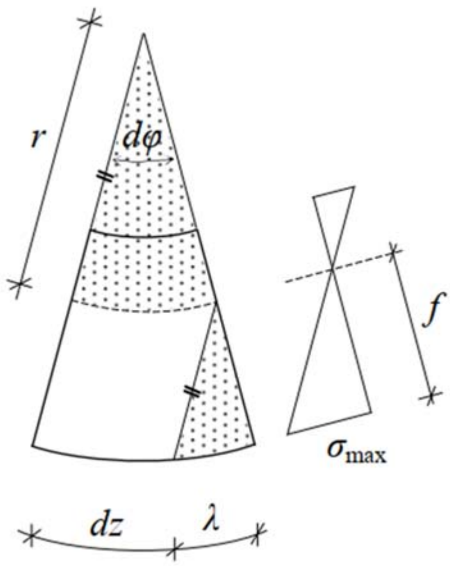
4.3 THE DIFFERENTIAL EQUATION OF AN ELASTIC CURVE

Deformation of a bent beam:



Elementary section during deformation:





Hooke's law applies:

$$\sigma = E * \varepsilon$$

Specific elongation in this case:

$$\varepsilon = \frac{\lambda}{dz}$$

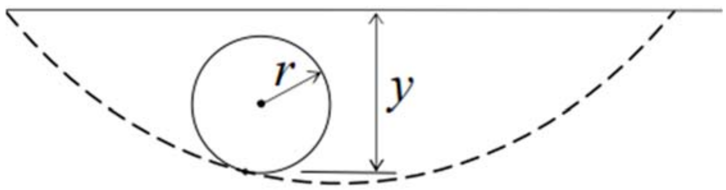
The stress in the outermost fiber:

$$\sigma = \frac{M}{I_x} f$$

Substituting these into the Hooke equation:

$$\sigma = \frac{M}{I_x} f = E * \varepsilon = E * \frac{\lambda}{dz} \longrightarrow \frac{M}{I_x} f = E * \frac{\lambda}{dz}$$

The radius of curvature of the beam (the radius of the tangent circle at the given location) is r, from which:



$$\frac{\lambda}{dz} = \frac{f}{r}$$

Thus, the previous equation is transformed:

$$\frac{M}{I_x} f = E * \frac{\lambda}{dz} \quad \longrightarrow \quad \frac{M}{E * I_x} = \frac{1}{r}$$

From the mathematical definition of the tangent circle:

$$\frac{1}{r} = \frac{y''}{\sqrt{\left(1 + y'^2\right)^3}}$$

Under our assumptions, the deformations are small, so that y and y', hence y'^2 <<1. For this reason, we can use the following approximation:

$$\sqrt{\left(1 + y'^2\right)^3} \cong 1; \text{ thus } \frac{1}{r} = y'';$$

Differential equation of the curved axis line:

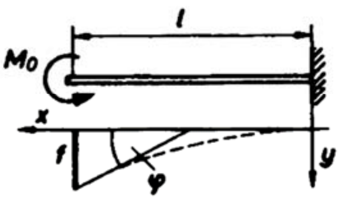
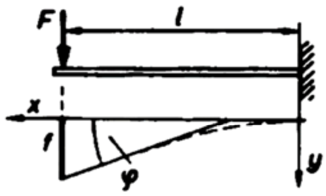
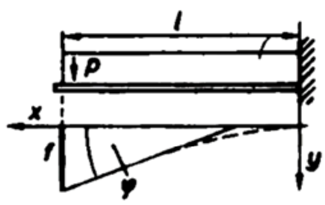
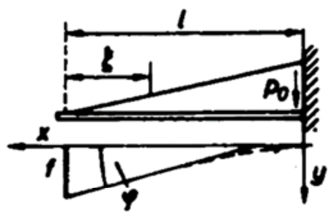
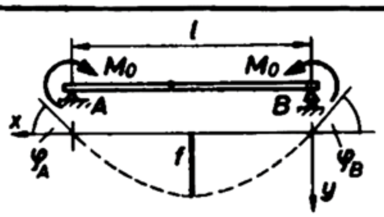
$$y''(z) = -\frac{M(z)}{E * I_x}$$

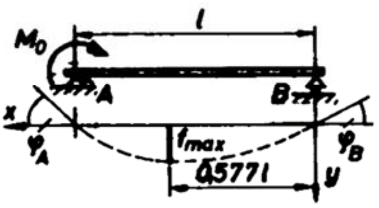
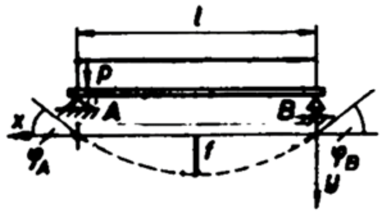
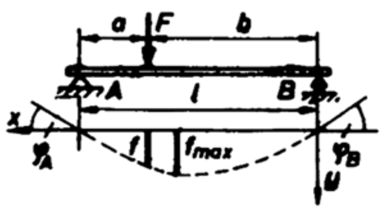
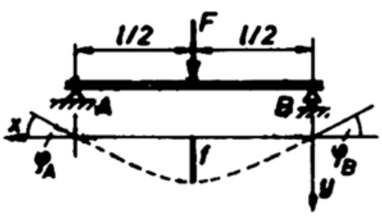
In the equation, y(z) is the function of the curved axis line, M(z) is the moment function, E is the elastic modulus of the material of the beam and Ix is the area moment of inertia of the cross section on the centre of gravity axis perpendicular to the plane of bending.

The general solution of the incomplete second-order differential equation can be obtained relatively easily by double integration.

4.4 DEFORMATION FORMULAE

Derived from the differential equation of an elastic curve for common constraints and loads.

Constraints and load of the beam, the approximate shape of the bent elastic curve.	Equation of the elastic curve and it's first derivative.	Translations and rotations
	$y = \frac{M_0}{2IE}x^2$ $y' = \frac{M_0}{IE}x$	$f = \frac{M_0 l^2}{2IE}$ $\varphi = \frac{M_0 l}{IE}$
	$y = \frac{F}{6IE}(3lx^2 - x^3)$ $y' = \frac{F}{2IE}(2lx - x^2)$	$f = \frac{Fl^3}{3IE}$ $\varphi = \frac{Fl^2}{2IE}$
	$y = \frac{p}{24IE}(6l^2x^2 - 4lx^3 + x^4)$ $y' = \frac{p}{6IE}(3l^2x - 3lx^2 + x^3)$	$f = \frac{pl^4}{8IE}$ $\varphi = \frac{pl^3}{6IE}$
	$y = \frac{p_0}{120IEl}(\xi^5 - 5l^4\xi + 4l^5)$ $y' = \frac{p_0}{24IEl}(\xi^4 - l^4); \quad \xi = l - x$	$f = \frac{p_0 l^4}{30IE}$ $\varphi = \frac{p_0 l^3}{24IE}$
	$y = \frac{M_0}{2IE}(lx - x^2)$ $y' = \frac{M_0}{2IE}(l - 2x)$	In the middle $f = \frac{M_0 l^2}{8IE}$ $\varphi_A = \varphi_B = \frac{M_0 l}{2IE}$

Constraints and load of the beam, the approximate shape of the bent elastic curve.	Equation of the elastic curve and it's first derivative.	Translations and rotations
	$y = \frac{M_0}{6IEl}(l^2x - x^3)$ $y' = \frac{M_0}{6IEl}(l^2 - 3x^2)$	$x_0 = 0{,}577l$ location $f_{max} = 0{,}0641\frac{M_0 l^2}{IE}$ $\varphi_B = \frac{M_0 l}{6IE}$ $\varphi_A = 2\varphi_B$
	$y = \frac{p}{24IE}(l^3x - 2lx^3 + x^4)$ $y' = \frac{p}{24IE}(l^3 - 6lx^2 + 4x^3)$	In the middle $f = \frac{5pl^4}{384IE}$ $\varphi_A = \varphi_B = \frac{pl^3}{24IE}$
	$0 \le x \le a$: $y = \frac{Fax}{6IEl}(l^2 - a^2 - x^2)$ $y' = \frac{Fa}{6IEl}(l^2 - a^2 - 3x^2)$ $a \le x \le l$: $y = \frac{Fb(l-x)}{6IEl}[l^2 - b^2 - (l-x)^2]$ $y' = \frac{Fb}{6IEl}[3(l-x)^2 - l^2 + b^2]$	$x = b$ location $f = \frac{Fa^2b^2}{3IEl}$ f_{max} location, if $b > a$ $x_0 = \sqrt{\frac{2ab + b^2}{3}}$
	$0 \le x \le l/2$: $y = \frac{F}{48IE}(3l^2x - 4x^3)$ $y' = \frac{F}{16IE}(l^2 - 4x^2)$	In the middle $f = \frac{Fl^3}{48IE}$ $\varphi_A = \varphi_B = \frac{Fl^2}{16IE}$

5 UNIDIRECTIONAL PURE LOADS

5.1 HOOKE'S LAW AND UNIDIRECTIONAL DEFORMATION

As the deformation is unidirectional, only elongation/compression is present.

STRESS STATE

$$\overline{\overline{F_P}} = \begin{bmatrix} \sigma_x & 0 & 0 \\ 0 & \sigma_y & 0 \\ 0 & 0 & \sigma_z \end{bmatrix}$$

or

$$\overline{\overline{F_P}} = \begin{bmatrix} 0 & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & 0 & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & 0 \end{bmatrix}$$

DEFORMATION STATE:

Something like

$$\underline{\underline{A}} = \begin{bmatrix} \varepsilon_x & 0 & 0 \\ 0 & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix}$$

or

$$\left[\underline{\underline{A}}\right]_{(R\varphi x)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}\gamma_{\varphi x} \\ 0 & \frac{1}{2}\gamma_{x\varphi} & 0 \end{bmatrix}$$

However, the stresses and deformations are not always present in all directions, in those places in the tensor can be also zeroes.

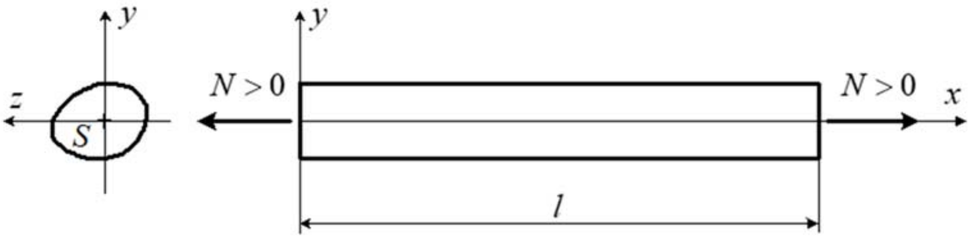
HOOKE'S LAW: The stress needed to cause deformation scales linearly with respect to that deformation.

$$\sigma_x = E * \varepsilon_x$$

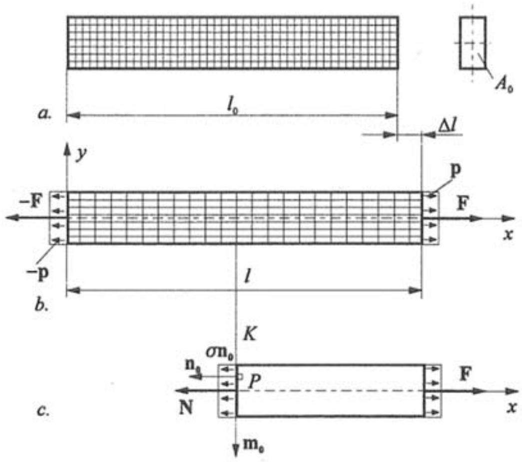
Where E is elastic modulus, a material property.

5.2 CENTRAL TENSION AND COMPRESSION

The force is acting in the centroid of the cross-section and parallel to it.



5.2.1 STRESS STATE:



$$\underline{\underline{F}} = \begin{bmatrix} \sigma_x & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \Rightarrow \sigma_x = \frac{N}{A}$$

The stress can be calculated by simply dividing the normal force with the section area.

5.2.2 HOOKE'S LAW AND DEFORMATION

LONGITUDINAL DEFORMATION:

$$\varepsilon_h = \varepsilon_x = \frac{l' - l}{l}$$

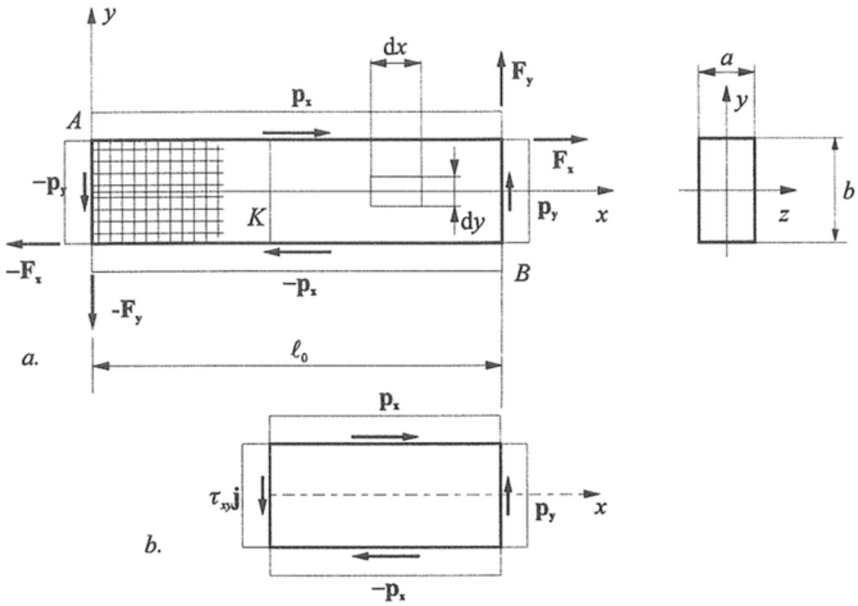
TRANSVERSAL DEFORMATION:

$$\varepsilon_k = \varepsilon_y = \varepsilon_z = -\nu \varepsilon_x$$

Where ν is Poisson's ratio, a material property.

5.3 PURE SHEAR

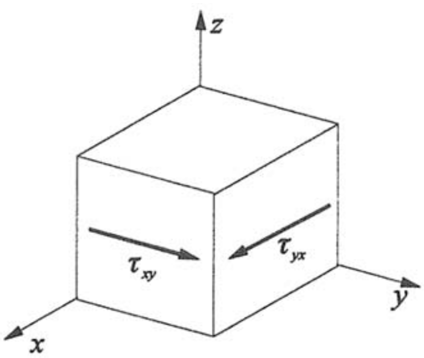
The force is acting perpendicular to the cross-section in a single line, thus not bending moment. In reality, there is no pure shear, it is always paired with bending, but there are some cases when the problem can be simplified and can be treated as if pure shearing was present.



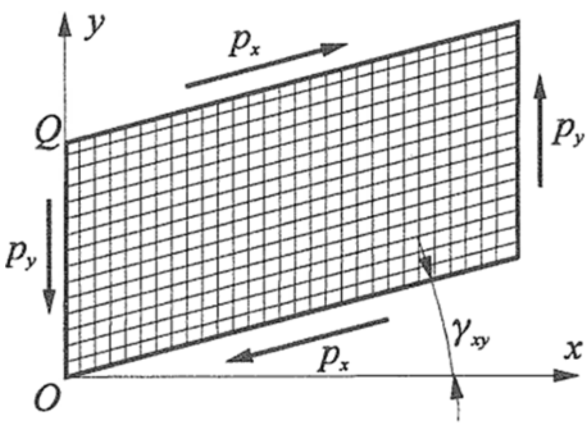
5.3.1 STRESS STATE:

$$\overline{\overline{F_P}} = \begin{bmatrix} 0 & \tau_{xy} & 0 \\ \tau_{yx} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\tau = \frac{F}{A}$$



5.3.2 HOOKE'S LAW AND SHEAR DEFORMATION



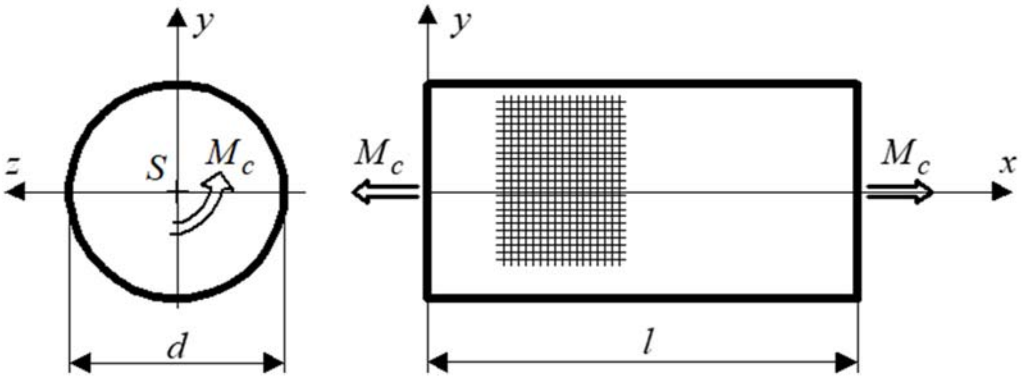
Hooke's law:

$$\tau_{xy} = G * \gamma_{xy}$$

Where G is shear modulus, a material property.

5.4 PURE TWIST

A moment vector is acting in the centroid of the cross-section and parallel to it.



5.4.1 STRESS STATE:

$$\left[\underline{\underline{F}}\right] = \begin{bmatrix} 0 & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & 0 & 0 \\ \tau_{zx} & 0 & 0 \end{bmatrix}$$

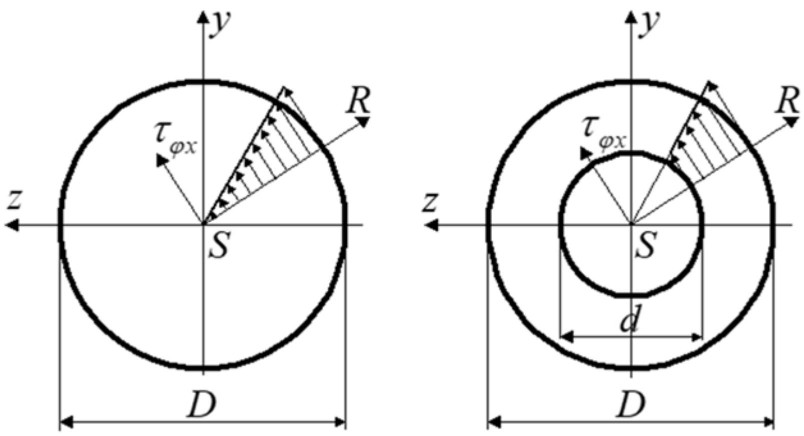
or

$$\underset{(R\varphi x)}{\left[\underline{\underline{F}}\right]} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \tau_{\varphi x} \\ 0 & \tau_{x\varphi} & 0 \end{bmatrix}$$

$$\tau_{xy} = \tau_{yx} = -\frac{M_c}{I_p} z$$

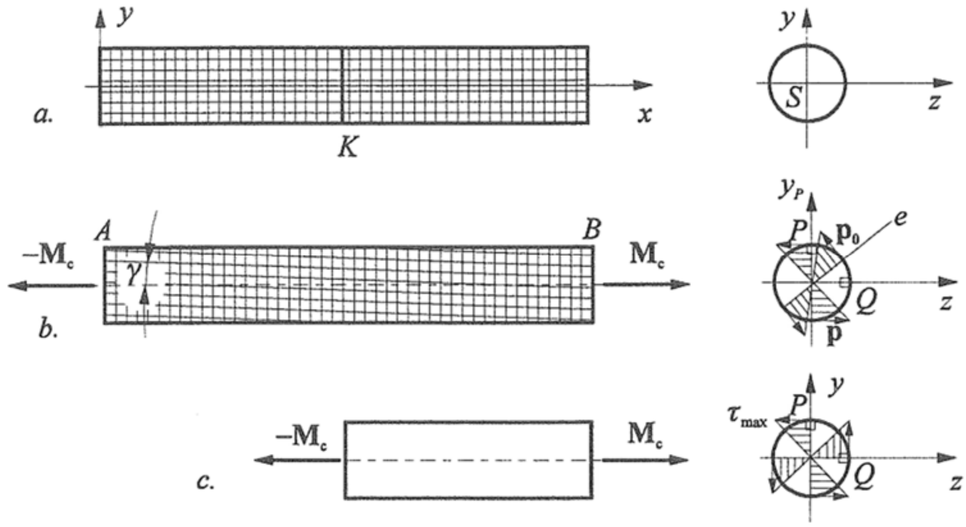
$$\tau_{xz} = \tau_{zx} = \frac{M_c}{I_p} y$$

STRESS DISTRIBUTION



Stress can only be where there is material – obviously.

5.4.2 TWIST DEFORMATION



DEFORMATION STATE:

$$\underset{(R\varphi x)}{\left[\underline{\underline{A}}\right]}=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & \frac{1}{2}\gamma_{\varphi x}\\ 0 & \frac{1}{2}\gamma_{x\varphi} & 0\end{bmatrix}$$

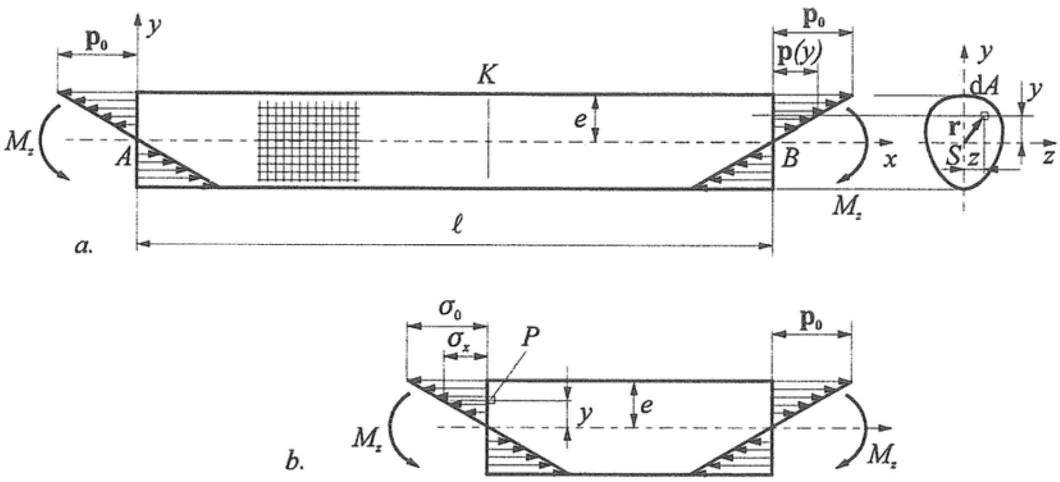
$$\varepsilon_R = \varepsilon_\varphi = \varepsilon_x = 0$$

$$\gamma_{R\varphi} = \gamma_{\varphi R} = 0, \quad \gamma_{xR} = \gamma_{Rx} = 0,$$

$$\gamma_{x\varphi} = \gamma_{\varphi x} = \gamma$$

5.5 PURE, STRAIGHT BENDING

A single moment is bending the whole beam.



5.5.1 STRESS STATE:

$$\left[\underline{\underline{F}}(y)\right]=\begin{bmatrix}\sigma_x(y) & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$\sigma_x = \frac{M}{I_z} e$$

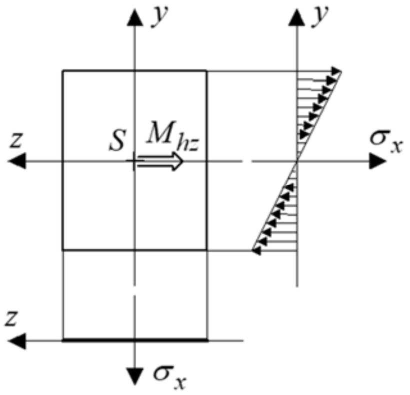
Where:

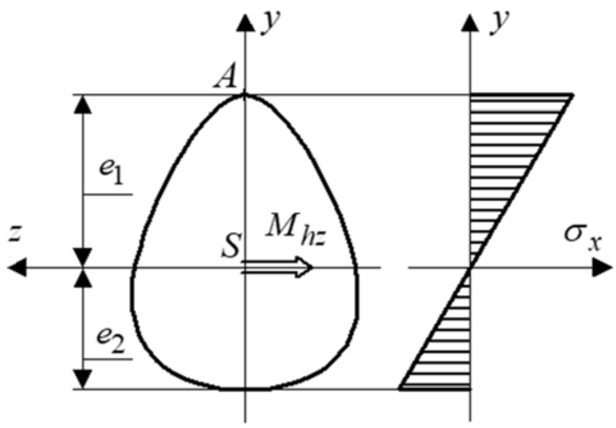
M: Bending moment

I_z: Moment of inertia on the axis of bending

e: Distance of the neutral axis from the dangerous point

STRESS DISTRIBUTION

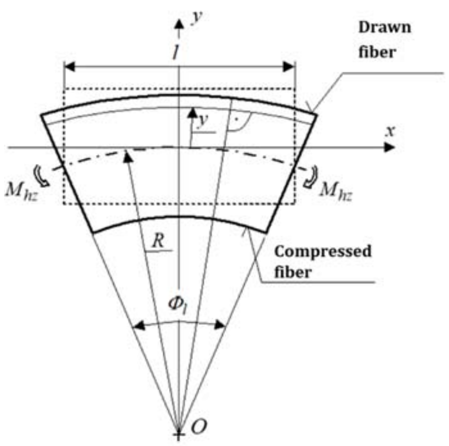


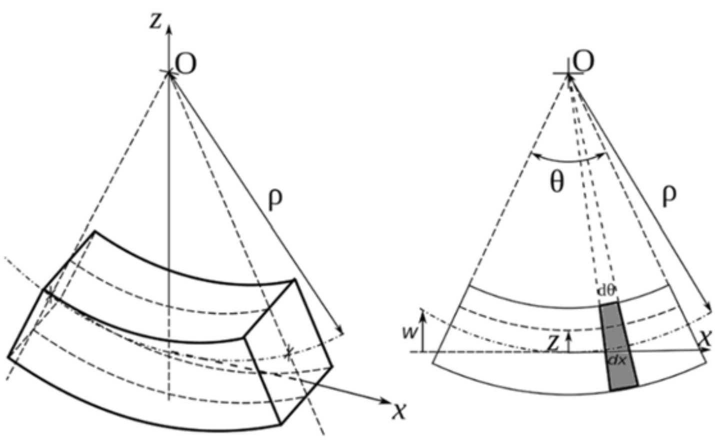


DANGEROUS POINT:

The point on the cross-section which is farthest from the neutral axis. The stress from bending is maximum at this point.

5.5.2 BENDING DEFORMATION





According to Euler–Bernoulli beam theory:

Each cross-section of the beam is planar and are at 90 degrees to the neutral axis.

DEFORMATION STATE

$$\left[\underline{\underline{A}}(y)\right] = \begin{bmatrix} \varepsilon_x & 0 & 0 \\ 0 & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix}$$

Simple Hooke's law applies at every point:

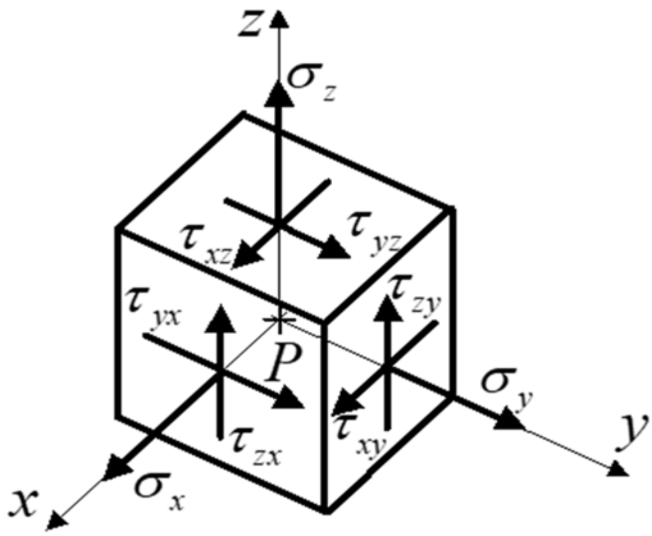
$$\sigma_x = E * \varepsilon_x$$

6 STRESS STATES

6.1 SPATIAL STRESS STATE

There are various ways to represent stress state.

6.1.1 ELEMENTARY UNIT CUBE



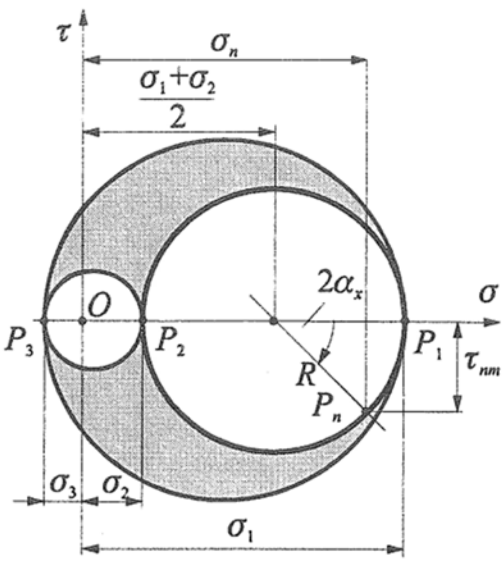
An infinitesimally small cube can be used to represent the stresses on three perpendicular planes.

6.1.2 STRESS TENSOR

As described earlier, the stress tensor represents stresses like so:

$$\overline{\overline{F_P}} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}$$

6.1.3 MOHR'S CIRCLE

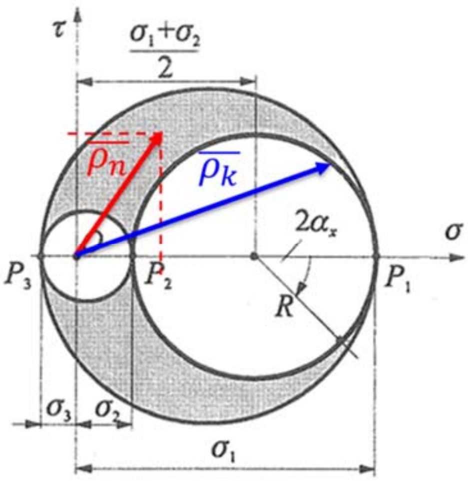


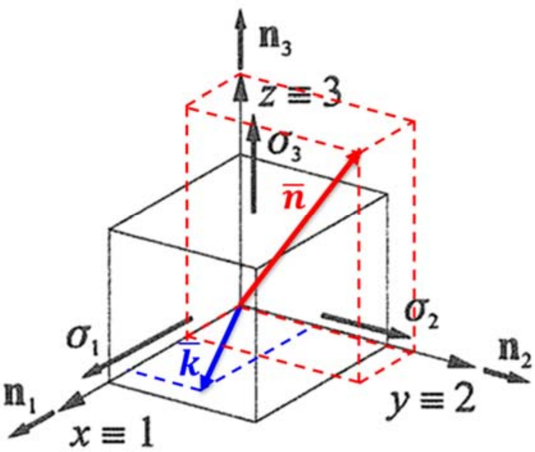
This representation is used to visualize the relationships between the normal and shear stresses acting on various inclined planes at a point in a stressed body.

Mohr's Circle can also be used to calculate principal stresses, maximum shear stresses and stresses on inclined planes.

The circle is named after its developer, German Civil Engineer Otto Mohr. (1835-1918)

The endpoints of the stress vectors for each direction in a σ-τ coordinate system fall within two arc triangles:





Where τ = 0, those directions are the principal directions.

They are mutually perpendicular.

DEFINITION: If $\bar{n}$ is a principal direction, then

$$\overline{\rho_n} = \sigma_n \bar{n}$$

Principal plane: in which one of the principal stresses is 0.

The endpoints of the stress vectors corresponding to the directions in the principal planes are located on the principal planes.

IT IS CONVENIENT TO DRAW A MOHR CIRCLE WHEN AT LEAST ONE PRINCIPAL DIRECTION IS KNOWN.

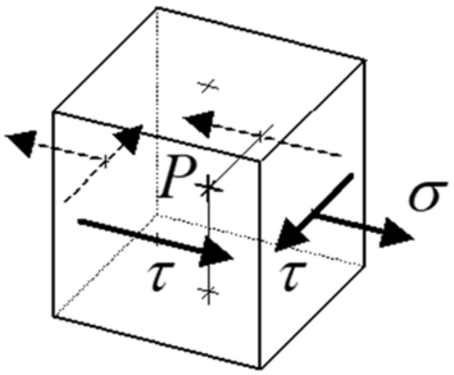
6.2 PLANAR STRESS STATE

For trusses, rods, the stress state is planar.

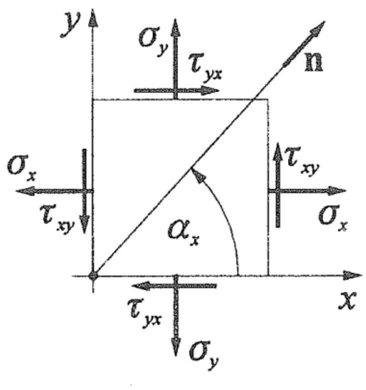
It is valid in all cases where there is only one normal and one shear stress (pair) in the stress tensor, and they fall in a plane.

$$\left[\underline{\underline{F}}_P\right] = \begin{bmatrix} \sigma_x & \tau_{xy} & 0 \\ \tau_{yx} & \sigma_y & 0 \\ 0 & 0 & \sigma_z \end{bmatrix}$$

For a planar voltage state, one of the principal stresses is always zero.



The spatial elementary unit can be converted to a simpler, planar version:



7 UNIDIRECTIONAL COMPLEX LOADS

1. Only tension (σ): Normal force + bending moment

   Inclined bending

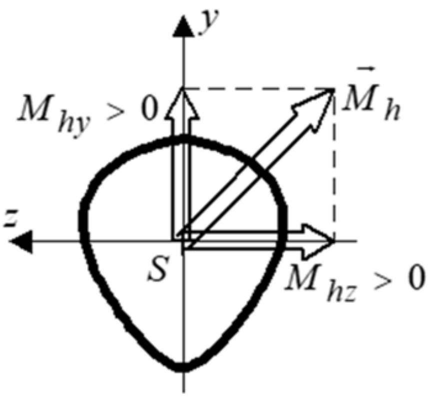
   Excentric tension/compression

2. Only shear (τ): Shearing force + twist

   Does not exist!

7.1 INCLINED BENDING

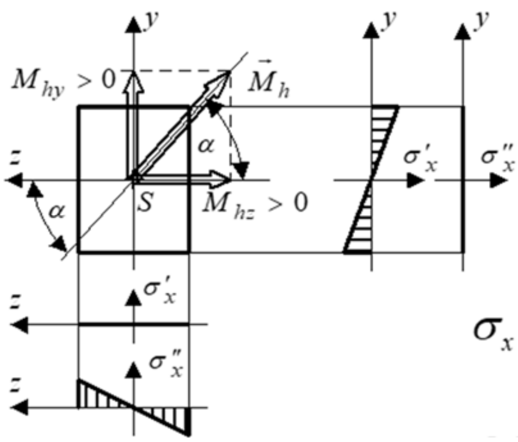
Definition: The moment vector is not parallel to any of the prime moment of inertia axes of the cross-section.



$$\vec{M}_h = M_{hy}\vec{j} - M_{hz}\vec{k}$$

Calculation: Simply the superposition of two straight bendings.

Stress state

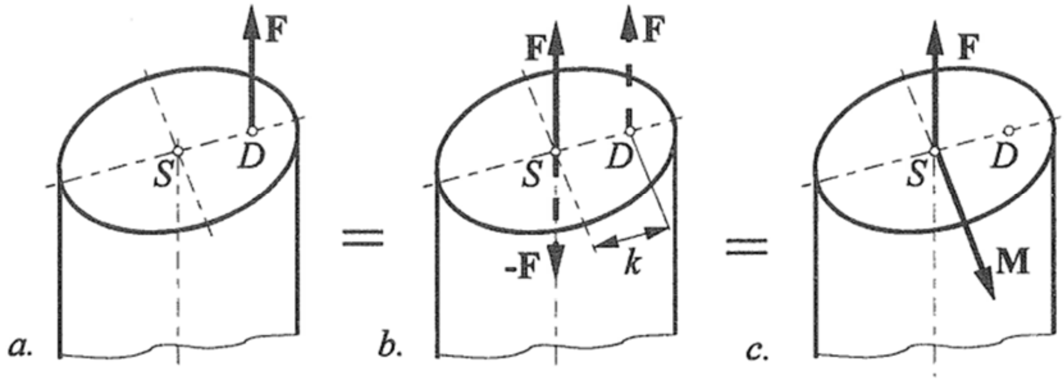


$$\underline{\underline{F}} = \begin{bmatrix} \sigma_x & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sigma_x = \sigma'_x + \sigma''_x = \frac{M_{hz}}{I_z} y + \frac{M_{hy}}{I_y} z$$

7.2 ECCENTRIC TENSION / COMPRESSION

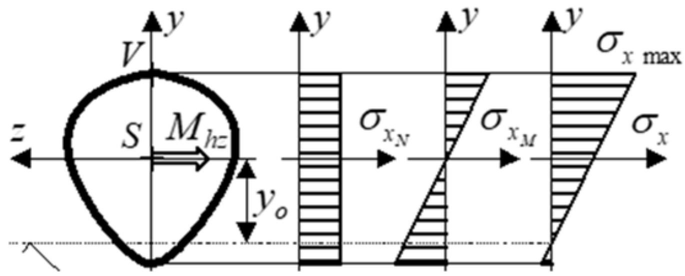
Definition: The load is eccentric when the resultant of the force system acting on the rod is parallel to the axis of the rod but not in the centroid of the cross-section.



Solution: Reduction of the eccentric load to the centroid. Result: moment and force.

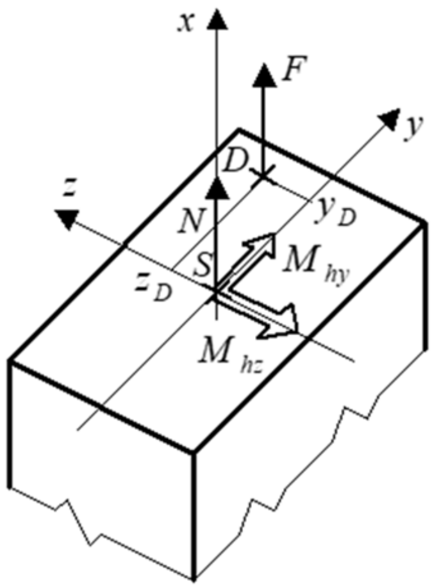
Stress state:

$$\underline{\underline{F}} = \begin{bmatrix} \sigma_x & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



Calculation: superposition of tensile stress from tension and bending

$$\sigma_x = \sigma_{x_N} + \sigma_{x_M} = \frac{N}{A} + \frac{M_{hz}}{I_z} y + \frac{M_{hy}}{I_y} z$$



8 STRESS THEORIES

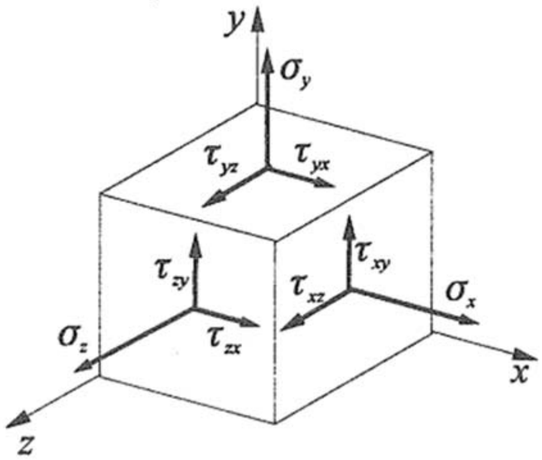
8.1 MULTIDIRECTIONAL COMPLEX LOADS

Simultaneous tensile-compressive (σ) and shear (τ) stresses

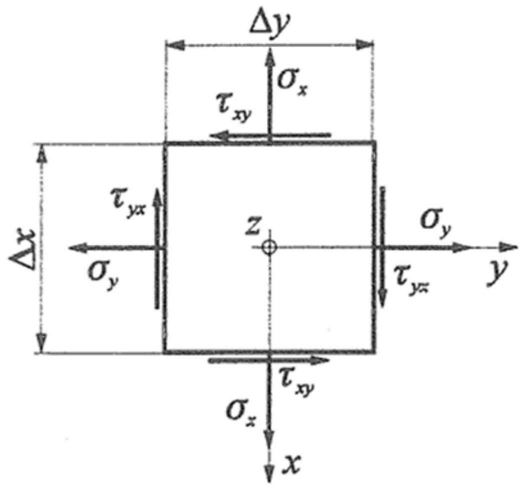
- Bending + shearing
- Bending + torsion
- Tension-compression + torsion
- Tension-compression + shearing
- Bending + shearing + tension
- Bending + shearing + torsion
- Bending + tension + torsion
- Tension + torsion + shearing
- Bending + tension + shearing + torsion

Superposition does not work. Reduced stress is required

Spatial stress state:



Planar stress state:



8.2 COULOMB'S THEORY

Failure occurs at a point in the material when the maximum normal stress at that point reaches the tensile or compressive failure strength.

According to Coulomb, the reduced stress is equal to the highest absolute value of the principal stresses.

$$\sigma_{red}\left(Coulomb\right)=\max\left(\left|\sigma_1\right|,\left|\sigma_3\right|\right)$$

For brittle materials, Coulomb's theory gives a good description of failure when there is a dominant principal stress, compared to which the other two principal stresses are less dangerous.

8.3 HUBER - VON MISES – HENCKY (HMH) THEORY

Two stress states are equally dangerous to failure if their deformation energies are equal.

$$\sigma_{red}\left(HMH\right)=\sqrt{\frac{1}{2}\left[\left(\sigma_x-\sigma_y\right)^2+\left(\sigma_y-\sigma_z\right)^2+\left(\sigma_x-\sigma_z\right)^2+6\left(\tau_{xy}^2+\tau_{yz}^2+\tau_{xz}^2\right)\right]}$$

$$\sigma_{red}\left(HMH\right)=\sqrt{\sigma_x^2+3\tau_{x\varphi}^2}$$

The HMH theory gives a good description of the occurrence of failure for ductile materials.

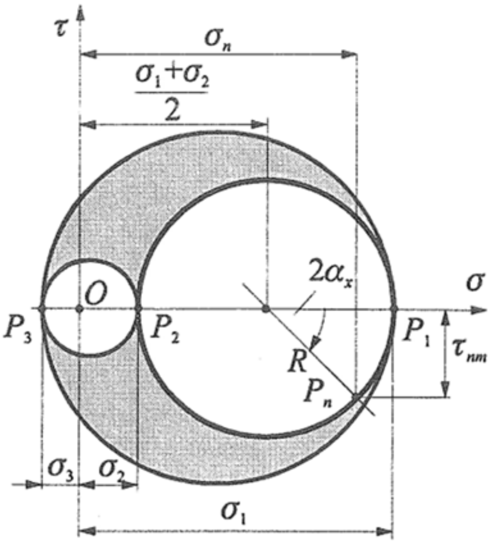
8.4 MOHR'S THEORY

Two general spatial stress states are equally dangerous for failure if the diameters of their corresponding maximum Mohr circles are equal.

According to Mohr, a stress state in a point is characterised by the diameter of the largest Mohr circle for damage.

$$\sigma_{red}\left(Mohr\right)=\sqrt{\sigma_x^2+4\tau_{x\varphi}^2}$$

Mohr's theory gives a good prediction of the occurrence of failure for ductile materials.



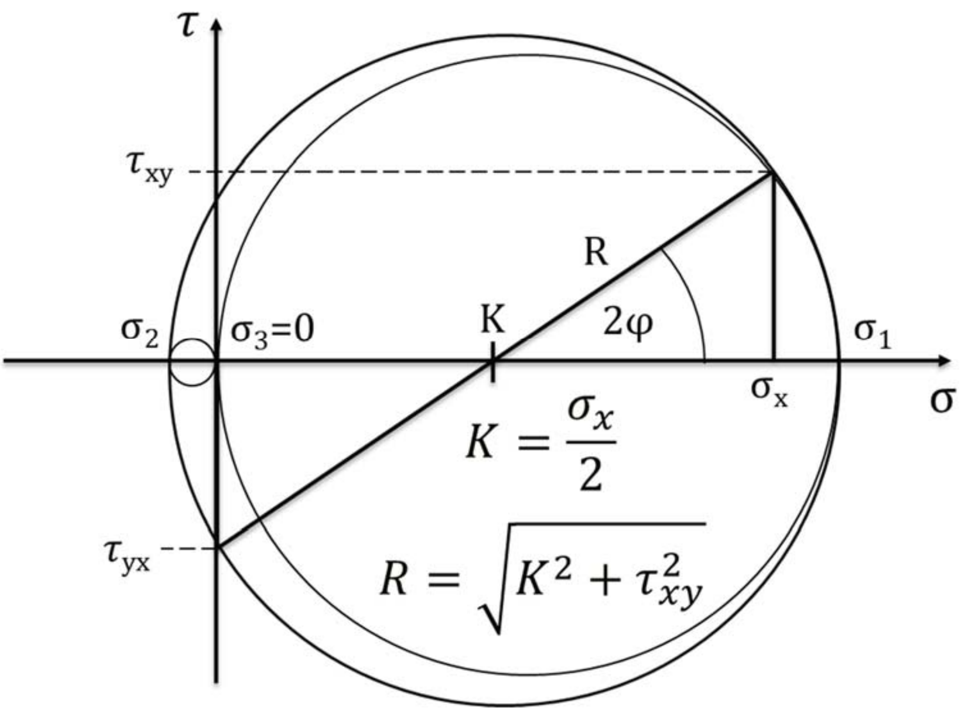
The reduced stress calculated by Mohr and HMH theory differ only slightly.

$$\sigma_{red}\left(HMH\right)\le\sigma_{red}\left(Mohr\right)$$

8.5 PLANAR PRINCIPAL STRESS

MOHR'S PLANAR STRESS CIRCLE DIAGRAM DRAWING PROCESS

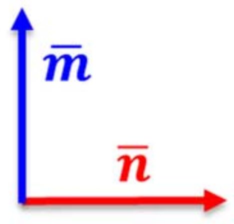
1. Draw a cartesian σ-τ coordinate system.
2. Measure σ_x on the horizontal axis.
3. Measure τ_{xy} at σ_x and τ_{yx} at 0.
4. Connect the obtained points, obtain the center and radius of circle 1-2.
5. We draw the MOHR circle 1-2.
6. If necessary for the calculation, draw circles 1-3 and 2-3.
7. Calculate the values of K and R. $K = \frac{\sigma_x}{2}$; $R = \sqrt{K^2 + \tau_{xy}^2}$;
8. From this we calculate σ_1, σ_2, and φ. $\sigma_1 = K + R$; $\sigma_2 = K - R$; $\tan 2\varphi = \frac{2\tau_{xy}}{\sigma_x}$;



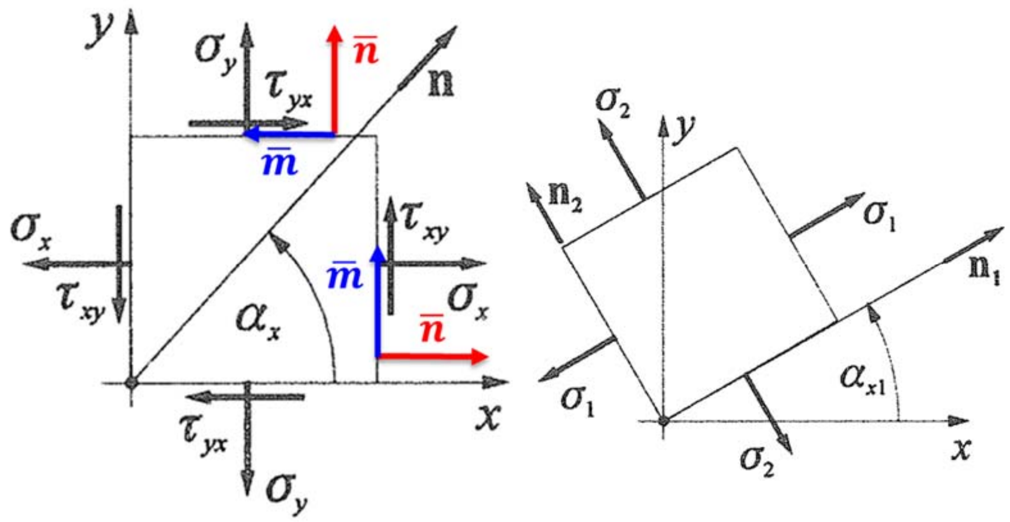
From these, the equations for the principal stresses can be obtained, but it is best not to use them, the circle is more reliable.

$$\sigma_1 = \frac{\sigma_x}{2} + \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_{xy}^2};\ \sigma_2 = \frac{\sigma_x}{2} - \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_{xy}^2}$$

The sign of τ is determined in the local right-handed n-m coordinate system.



n is perpendicular to the plane under consideration, and the sign of τ is shown on m.



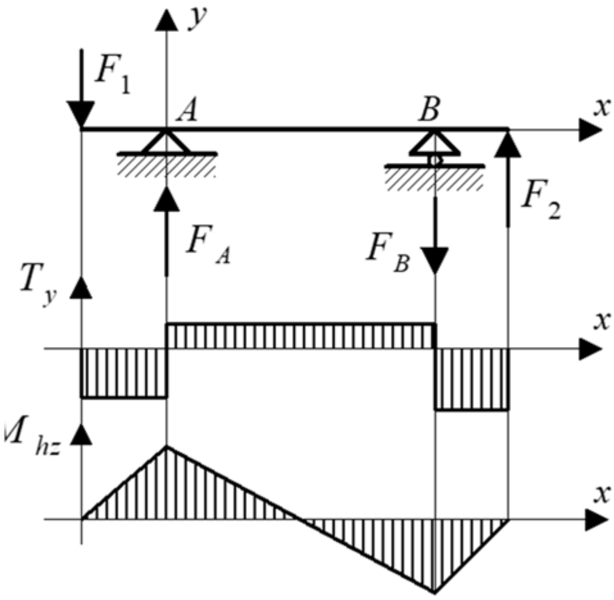
With this method, the sign of τ can be determined in any position or rotation of the elementary unit.

9 MULTIDIRECTIONAL COMPLEX LOADS

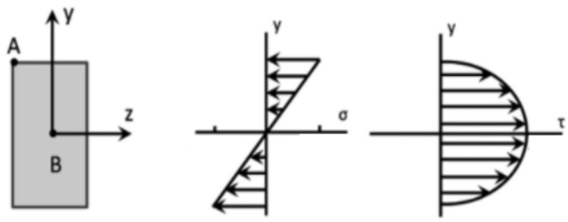
9.1 SIMULTANEOUS BENDING AND SHEARING

In reality, shearing is always paired with bending.

Example:



Stress distribution:



Stress state:

$$\left[\underline{\underline{F}}\right] = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & 0 & 0 \\ \tau_{zx} & 0 & 0 \end{bmatrix}$$

Calculation of the stresses:

$$\sigma_x = \frac{M_{hz}}{I_z} y \quad \text{and} \quad \tau_{yx} = -\frac{T_y S_{1z}(y)}{I_z a(y)}$$

Where:

T_y - shear force

M_{hz} - bending moment

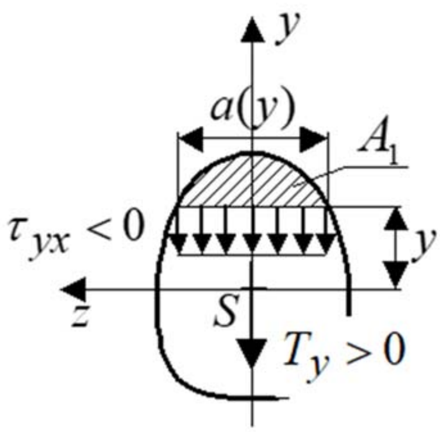
I_z – second moment of area of the cross-section calculated in the z-axis

a - x dimension of the cross-section at the point of inspection.

S_{1z} - static moment of the area of the cross-section above the point of inspection calculated in the z axis.

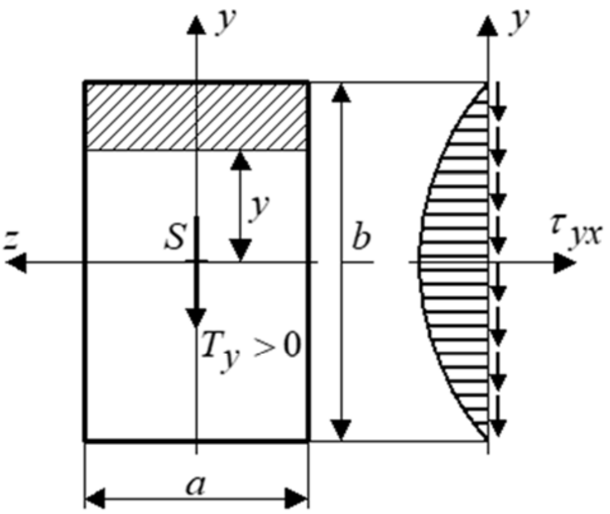
STATIC MOMENT OF SHEAR

The static moment of the area of the cross-section above the point of inspection in the z axis which was used in the formula of the shear stress is calculated like so:



The area of the cross-section above the point of inspection is multiplied by the centroid distance from z axis, thus obtaining the static moment of that area on the z axis.

E.g. rectangle:



$$a(y) = a,$$

$$I_z = \frac{ab^3}{12},$$

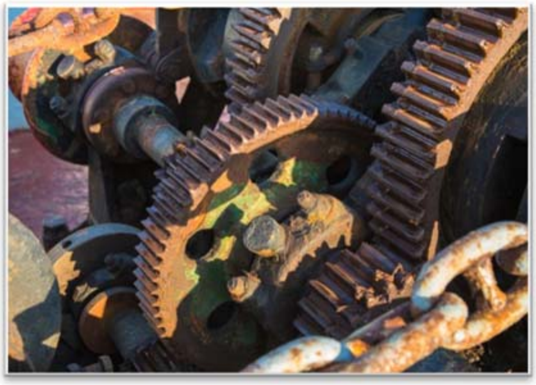
$$S_{1z}(y) = \underbrace{a\left(\frac{b}{2} - y\right)}_{A_{sraff}} \frac{1}{2}\left(\frac{b}{2} + y\right) = \frac{a}{2}\left(\frac{b^2}{4} - y^2\right)$$

$$\tau_{yx} = \tau_{yx}(y) = -\frac{T_y}{A} 6\left(\frac{1}{4} - \frac{y^2}{b^2}\right)$$

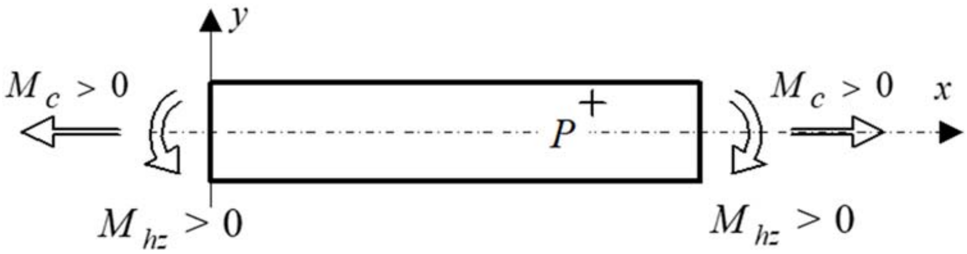
So, the stress distribution curve is a parabola of second degree.

9.2 SIMULTANEOUS BENDING AND TORSION

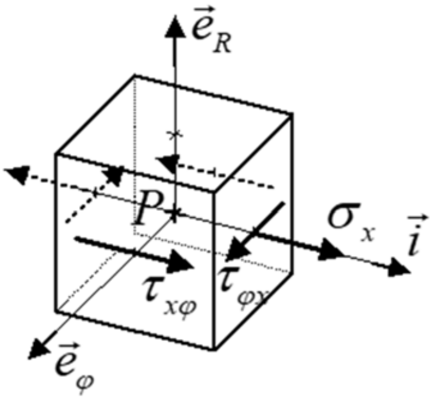
This is a very frequent complex load. Example: axis of any gear.



The beam is loaded by two perpendicular moments:



Stress state:

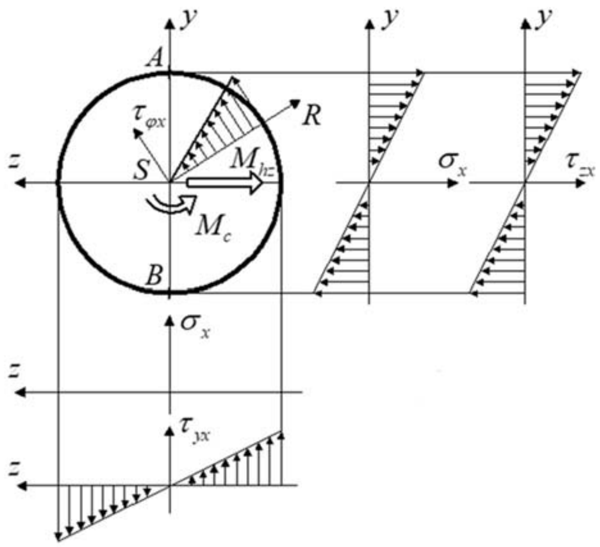


$$\underset{(R\varphi x)}{\left[\underline{\underline{F}}\right]} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \tau_{\varphi x} \\ 0 & \tau_{x\varphi} & \sigma_x \end{bmatrix}$$

Calculation of the stresses:

$$\sigma_x = \frac{M_{hz}}{I_z} y \quad \text{and} \quad \tau_{\varphi x} = \frac{M_c}{I_p} R$$

Stress distribution:



Stresses in the dangerous points A and B:

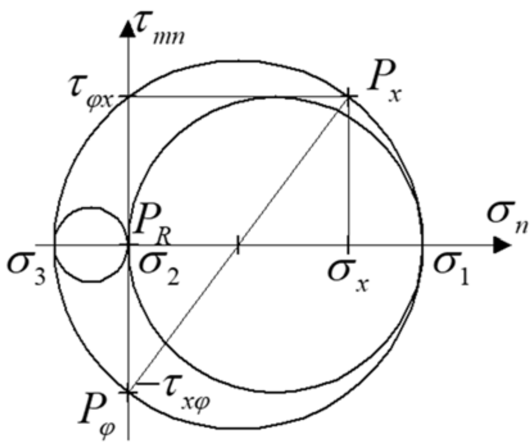
$$\sigma_{x\,\max} = \frac{|M_h|}{I_z}\frac{d}{2} = \frac{|M_h|}{K_z}$$

$$\tau_{x\varphi\,\max} = \frac{|M_c|}{I_p}\frac{d}{2} = \frac{|M_c|}{K_p}$$

$$I_p = 2I_z \;\Rightarrow\; K_p = 2K_z$$

Now the calculation of the reduced stress is required in the dangerous points.

Mohr’s circle at any point P of the beam, for positive normal stress:



Reduced stresses at the dangerous points:

$$\sigma_{red\,\max}(Coulomb) = \max\left(|\sigma_1|, |\sigma_3|\right)\Big|_{A,B}$$

$$\sigma_{red} = \sqrt{\sigma_x^2 + \beta\tau_{x\varphi}^2}$$

β=4 according to Mohr.

β=3 according to HMH.

Thus, the maximum reduced stress:

$$\sigma_{red\,\max} = \sqrt{\left(\frac{M_h}{K_z}\right)^2 + \frac{\beta}{4}\left(\frac{M_c}{K_z}\right)^2} = \frac{M_{red}}{K_z}$$

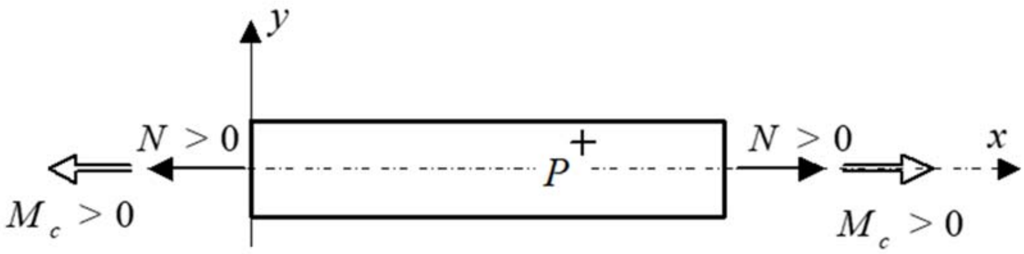
From these, we can introduce reduced moment:

$$M_{red} = \sqrt{M_h^2 + \frac{\beta}{4}M_c^2}$$

9.3 SIMULTANEOUS TENSION-COMPRESSION AND TORSION

The case when the structure is twisted about and pushed/pulled along its axis. Example: drilling.

There is a normal force and a moment vector acting in the centroid of the cross-section and parallel to it.



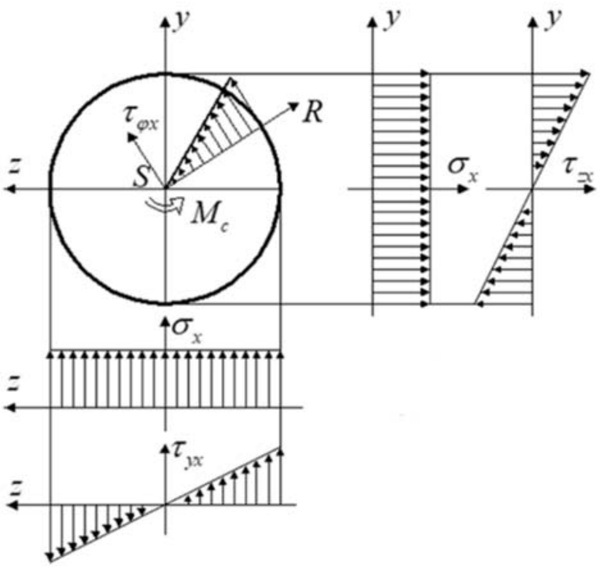
Stress state:

$$\underset{(R\varphi x)}{\left[\underline{\underline{F}}\right]} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \tau_{\varphi x} \\ 0 & \tau_{x\varphi} & \sigma_x \end{bmatrix}$$

Stress calculation:

$$\sigma_x = \frac{N}{A}, \qquad \tau_{x\varphi} = \frac{M_C}{I_p} R$$

Stress distribution:



Dangerous points are all the points on the perimeter of the cross-section circle.

Reduced stresses at the dangerous points:

$$\sigma_{red\,\max}(Coulomb) = \max\left(|\sigma_1|, |\sigma_3|\right)\Big|_{A,B}$$

$$\sigma_{red} = \sqrt{\sigma_x^2 + \beta\tau_{x\varphi}^2}$$

β=4 according to Mohr.

β=3 according to HMH.

DIMENSIONING METHOD IN THE CASE OF SIMULTANEOUS TENSION-COMPRESSION AND TORSION:

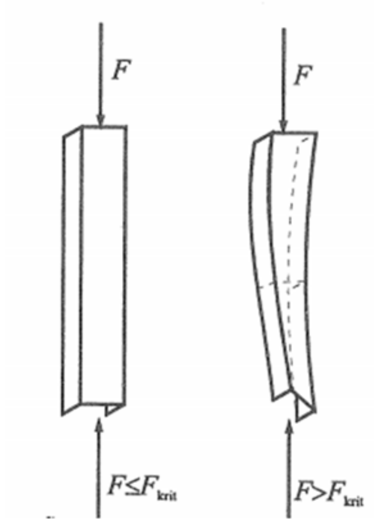
A simple iteration procedure:

1. We dimension for torsion, neglecting tension for the moment.
2. Check the selected (standard) sizing of the beam for simultaneous tensile and torsional loads.
3. If the beam does not fit, a larger standard size is selected.
4. Repeat steps 2-3 as necessary.

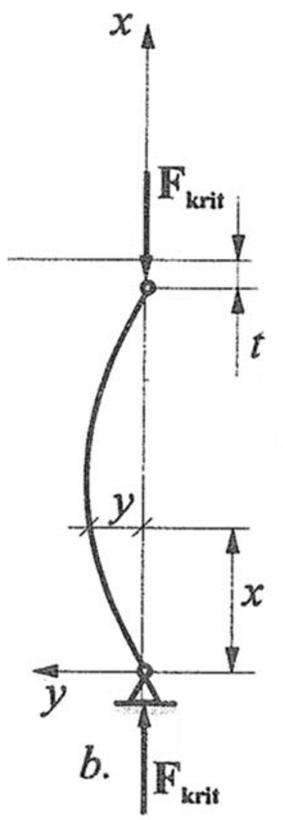
10 CENTRALLY PRESSED SLENDER STRUCTURES

Buckling:

- A loss of stability
- After buckling high values of deformation occurs as the result of relatively low force increase



10.1 ELASTIC BUCKLING



The moment at any point on the beam with an x coordinate:

$M = y^*F_{krit}$

The connection between the moment and deformation in y direction is the differential equation of an elastic curve:

$$y'' = -\frac{M(x)}{E \cdot I_2} = \frac{F_{krit}}{E \cdot I_2} \cdot y \xrightarrow{reordered} y'' + \frac{F_{krit}}{E \cdot I_2} \cdot y = 0$$

Where I_2 is the smallest moment of inertia of the cross-section.

By introducing $\alpha^2 = \frac{F_{krit}}{E \cdot I_2}$; we get $y'' + \alpha^2 y = 0$; Euler's differential equation.

This equation is a linear, homogeneous differential equation of the second order with constant coefficients.

The general solution of Euler's $y'' + \alpha^2 y = 0$; is

$$\boldsymbol{y = A \cdot sin(\alpha x) + B\,cos(\alpha x)}\,;$$

A and B can be determined from the boundary conditions.

1. At **x = 0; y= 0**, as the end of the beam does not move in y direction.
2. At **x = L; y= 0,** as the other end of the beam does not move in y direction either.

The value of y in the equation $y = A \cdot sin(\alpha x) + B\,cos(\alpha x)$ can be 0 only if B is 0, as cos(0) is 1, and not 0.

The other part, $A \cdot sin(\alpha x)$ can be 0 if A is 0, but then there is no buckling. If A is not 0, then $sin(\alpha x)$ has to be.

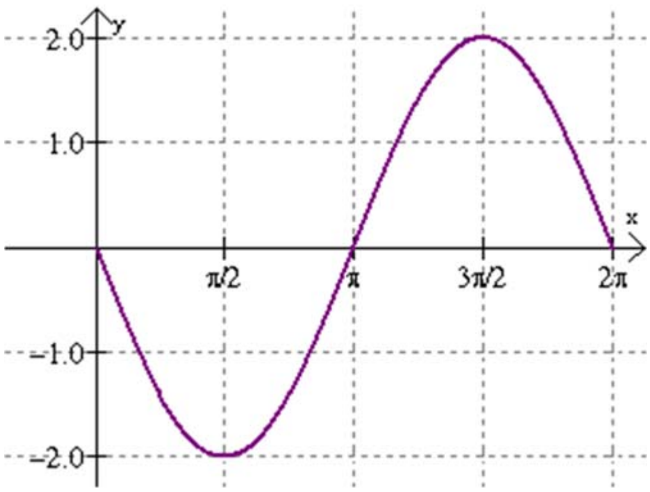
In conclusion, we are looking for the solutions of $sin(\alpha x) = 0$.

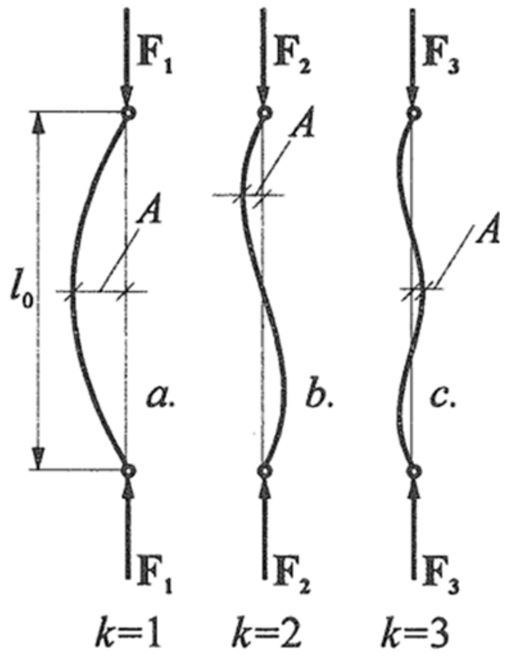
$sin(\alpha x) = 0$ if $\alpha x = \mathrm{k} * \pi$

where k is a positive integer.

As x = L; the equation is: $sin(\alpha L) = 0$.

Thus: $\boldsymbol{\alpha L} = \mathbf{k} * \boldsymbol{\pi}$





Introducing L_0 as the length of a half sine wave of the buckling beam, where k=1:

$\boldsymbol{\alpha L_0} = \boldsymbol{\pi}$; therefore $\alpha = \frac{\pi}{L_0}$ and $\alpha^2 = \frac{\pi^2}{L_0^2}$ and remember that $\alpha^2 = \frac{F_{crit}}{E \cdot I_2}$

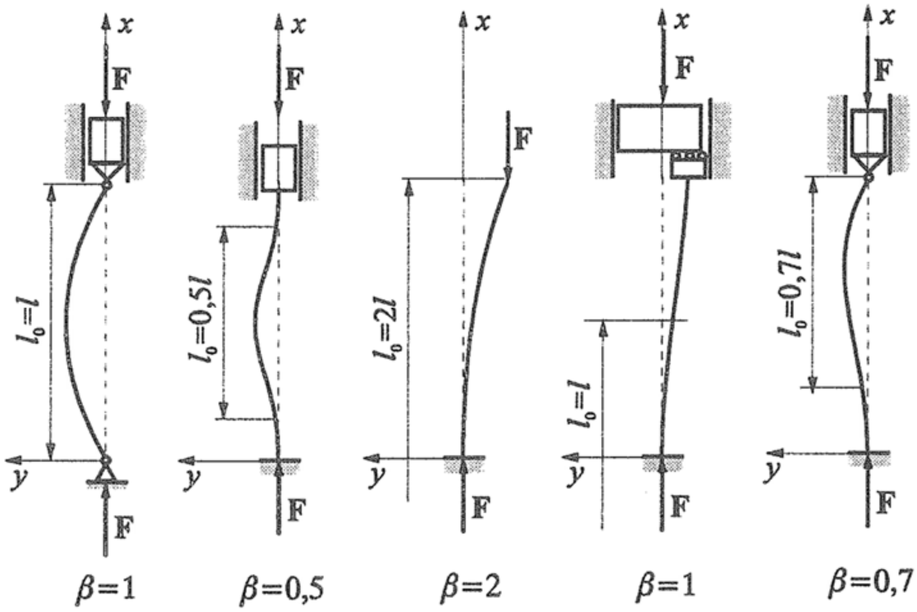
Finally we get Euler's formula for flexible buckling: $F_{crit} = \frac{\pi^2 \cdot E \cdot I_2}{L_0^2}$

And the critical stress is: $\sigma_{crit} = \frac{F_{krit}}{A} = \frac{\pi^2 \cdot E \cdot I_2}{\mathrm{A} \cdot L_0^2}$

Calculation of buckling length: $L_0 = \beta * L$

Where β depends on the end constraints of the beam.

β values by end constraints to determine buckling length:



$$L_0 = \beta * L$$

$$F_{crit} = \frac{\pi^2 \cdot E \cdot I_2}{L_0^2}$$

10.2 PLASTIC BUCKLING

Euler's formula only works in the material's flexible region.

Elastic buckling: high slenderness

Plastic buckling: low slenderness

10.2.1 DETERMINING THE TYPE OF BUCKLING

Introducing λ , the slenderness ratio:

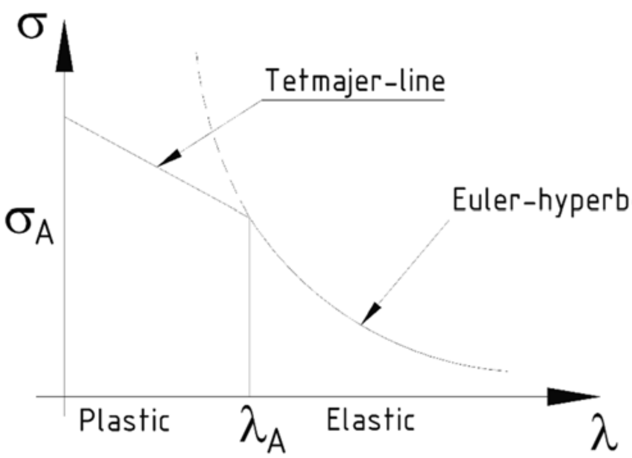
$$\lambda = \frac{L_0}{i_2}$$

Where i_2 is the lesser radius of gyration:

$$i_2 = \sqrt{\frac{I_2}{A}}$$

The slenderness ratio separating plastic and elastic buckling regions:

$$\lambda_A = \pi\sqrt{\frac{E}{\sigma_A}}$$



λ_A is a material property. Examples:

Steel: ~105

Wood: ~100

Cast iron: ~80

If the slenderness ratio is lower than λ_A the buckling is plastic, and Tetmajer's formula is used to calculate the critical stress:

$$\sigma_{crit} = a - b * \lambda$$

a and *b* are also material properties. Examples:

Steel: $\sigma_{crit} =$ (310-1,14 λ) MPa;

Wood: $\sigma_{crit} =$ (29,3-0,194λ) MPa;

Cast iron: $\sigma_{crit} =$(776-12λ+0,053λ^2) MPa;

11 SPRINGS

As springs deform under load, they store mechanical work in the form of deformation energy and then convert it back into mechanical work. The linear relationship between the deformation and the force that causes the deformation makes the spring suitable for force measurement as well.

Springs can be very diverse in their function, shape, and behaviour. The choice of material and the sizing must consider the stresses to which the spring will be subjected. For springs where the load is very frequently alternating (e.g., engine valve spring), the allowable stress should be kept below the fatigue limit to avoid fatigue failure.

11.1 RELATIONSHIPS THAT CHARACTERISE THE SPRINGS

1. The relationship between the deformation and the loading force

$$f = f\,(F)$$

2. The relationship between the loading force and the stress:

$$\sigma = \sigma(F)$$

3. The relationship between deformation and stress:

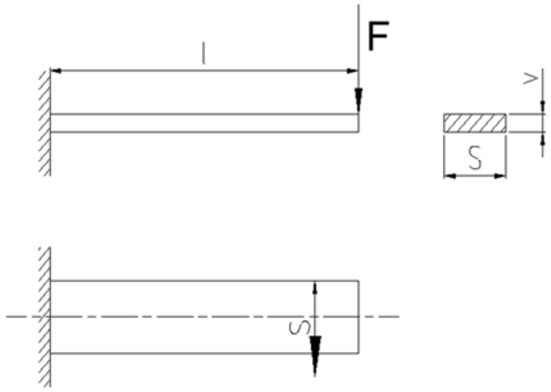
$$f = f\,(\sigma)$$

4. The amount of stored deformation work gives an indication of the degree of utilisation of the spring.

SPRINGS BY TYPE OF DEFORMATION:

1. contracted / tensioned springs.
2. torsion springs.
3. bent springs.

11.2 LEAF SPRING



The straight leaf spring is bent type, and can be dimensioned using the relationships already described.

The inspected spring is a cantilever with a rectangular cross-section of dimension s x v.

The relationship between deformation and force.

$$f = \frac{Fl^3}{3IE}; \qquad \text{where: } I = \frac{sv^3}{12}$$

Calculation of the maximum stress of the bent ring at the constraint:

$$\sigma = \frac{M}{I}e;$$

The maximum moment is $M = Fl$, and *e* is half of thickness *v*, thus

$$\sigma = \frac{Fl}{I} \cdot \frac{v}{2};$$

Reordered: $$\frac{F}{I} = \frac{2 \cdot \sigma}{l \cdot v} - t$$

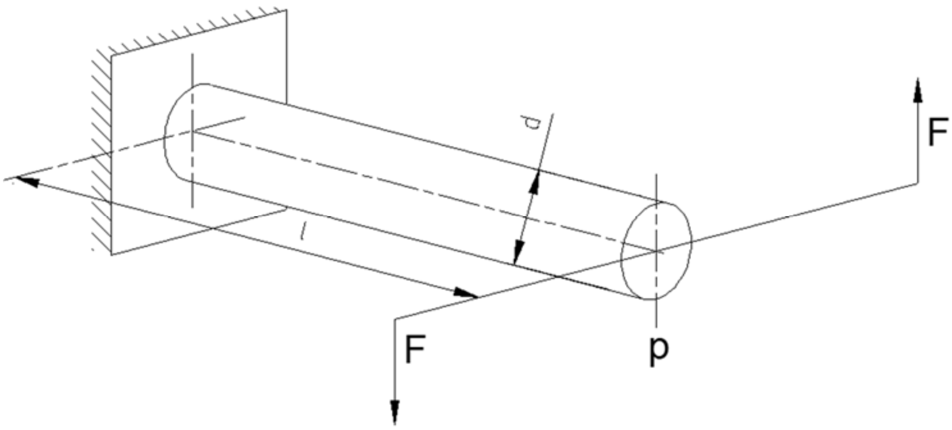
We can substitute the above equation in $f = \frac{Fl^3}{3IE}$; obtaining the formula for deformation in the factor of stress:

$$f = \frac{2\sigma}{lv} \cdot \frac{l^3}{3E} = \frac{2}{3} \cdot \frac{l^2}{v} \cdot \frac{\sigma}{E}$$

All three formulas are suitable for calculation.

11.3 TORSION SPRING

The end of the straight, cylindrical rod is twisted by $M_t = F \cdot p$ moment.



The polar second moment of area of the cross section:

$$I_p = \frac{d^4\pi}{32}$$

The rotation angle of the end of the rod in relation to the constraint:

$$\varphi = \frac{M_t l}{I_p G}$$

The moment from this equation:

$$M_t = \frac{\varphi I_p G}{l} = F \cdot p$$

Shear stress due to the moment:

$$\tau = \frac{M_t}{I_p} \cdot \frac{d}{2}$$

Substitute the value of M_t here, then:

$$\tau = \frac{\varphi I_p}{l I_p} \cdot \frac{d}{2} = \frac{\varphi \cdot G \cdot d}{2l}$$

Thus:

$$\varphi = 2\frac{\tau}{G} \cdot \frac{l}{d}$$

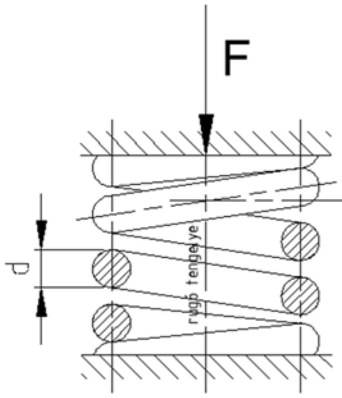
And the work accumulated in the torsion spring:

$$L = \frac{1}{2} M_t \varphi = \frac{1}{4} \frac{\tau^2}{G} V$$

where $V = \frac{d^2\pi}{4} l$ is the volume of the spring.

11.4 HELICAL COIL SPRING

The coil spring is perhaps the most commonly used type of spring. The spring is loaded by tension or compression at its ends in the direction of the cylinder axis. In the case of a tensioned spring, the threads move away from each other and the pitch increases), in the case of a compressed spring, the threads move closer together and the pitch decreases).



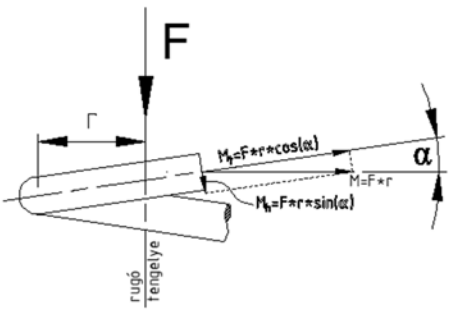
All the cross-sections of the spring are loaded by a moment

M = F*r.

The moment vector can be decomposed into two components:

M_h parallel to the plane of the cross-section

M_t perpendicular to the plane of the cross-section.



If the angle of attack α is small:

sin α ≈0 and M_h = Fr·sin(α) ≈ 0,

the bending stress can be neglected.

Since cos α ≈ 1, the torsion moment is:

M_t = Fr·cos(α) ≈ Fr

Accumulated deformation energy in the spring:

$$L = \frac{1}{2I_pG}\int_0^l M_t^2\,ds$$

$$M_t = Fr, \qquad M_t^2 = F^2\,r^2$$

Sbustituted:

$$L = \frac{1}{2I_pG}\int_0^l F^2r^2ds = \frac{1}{2I_pG}\,F^2r^2l$$

The displacement is obtained by Castigliano's theorem by differentiating L by F:

$$f = \frac{\partial L}{\partial F} = \frac{1}{2 I_p G} 2\, F\, r^2 l = \frac{F r^2 l}{I_p G}$$

The shear stress in the cross-section:

$$\tau = \frac{M_t}{I_p} \cdot \frac{d}{2} = \frac{Fr}{I_p} \cdot \frac{d}{2}$$

Thus:

$$Fr = \frac{2 \tau I_p}{d}$$

Substitute this into the displacement equation:

$$f = \frac{F r^2 l}{I_p G} = \frac{2 I_p r l}{d I_p G}$$

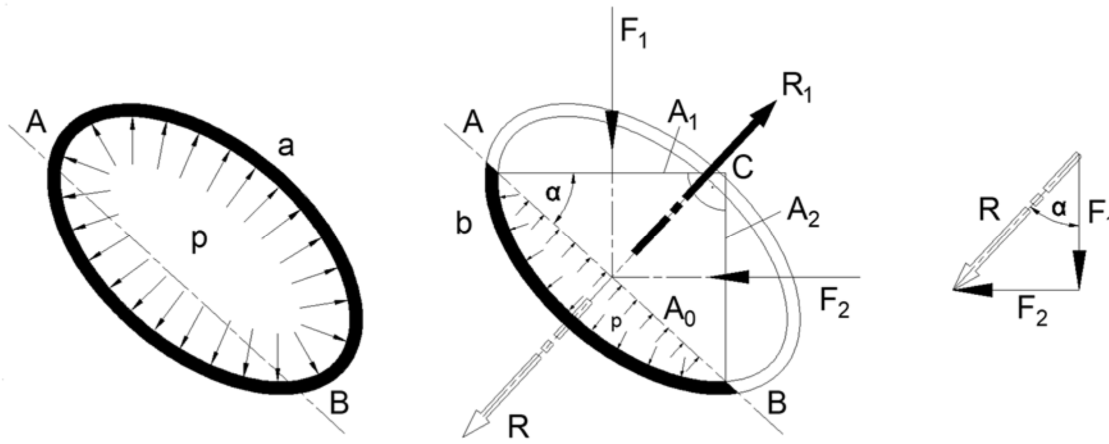
$$f = 2 \frac{\tau}{G} \cdot \frac{r}{d} l$$

In practice, the r/d ratio 3 ~ 5 is the usual range.

12 PIPES, VESSELS

12.1 PRESSURE OF A FLUID ON THE WALL OF AN ARBITRARY SHAPE VESSEL

The weight of the fluid resting in a vessel of arbitrary shape is neglected; the pressure p of the fluid is perpendicular to the vessel wall and its magnitude is constant. If the vessel is divided into two parts by a plane AB, both parts are in equilibrium.



The resultant of the pressure acting on AB planes surface of area A_0:

$$R_1 = p \cdot A_o$$

The resultant of the pressure R acting on the surface of curve AB is in equilibrium with R_1:

$$R = R_1 = p \cdot A_o$$

Take the plane AC of area A_1, enclosing an arbitrary angle α with the plane AB, and the plane BC of area A_2, perpendicular to it.

The resultant of the fluid pressure on the AC plane:

$$F_1 = p \cdot A_1$$

and the resultant of the fluid pressure on the BC plane:

$$F_2 = p \cdot A_2$$

Since $A_1 = A_0 \cdot \cos\alpha;$ and $A_2 = A_0 \cdot \sin\alpha;$

$F_1 = p \cdot A_0 \cdot \cos\alpha = R \cdot \cos\alpha;$ and $F_2 = p \cdot A_0 \cdot \sin\alpha = R \cdot \sin\alpha$

The same result is obtained from the vector triangle:

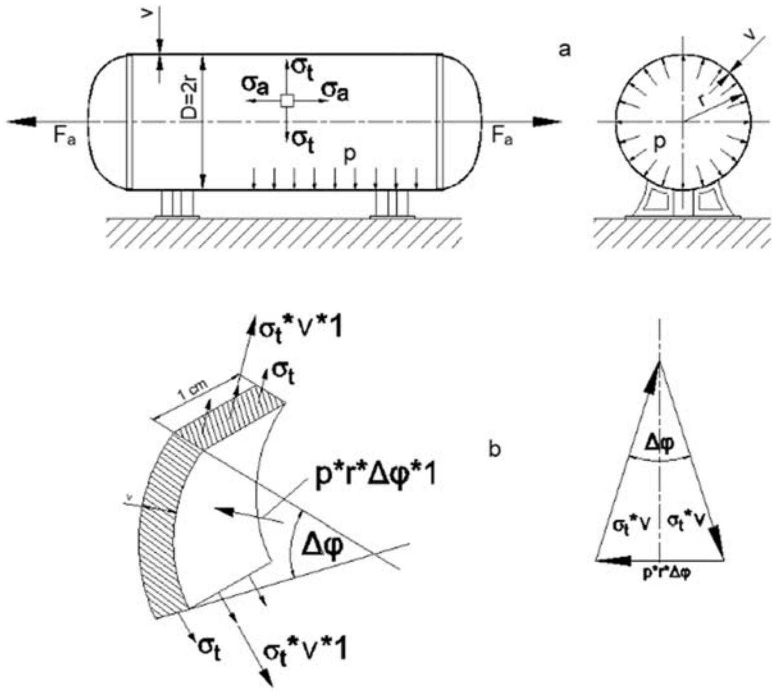
$F_1 = R \cdot \cos\alpha;$ and $F_2 = R \cdot \sin\alpha;$

Thus, the projection of the resultant R of the pressures acting on the surface of AB curve in a given direction is equal to the pressure of the fluid on the projection of the surface AB perpendicular to that direction.

The theorem is equally valid for liquid, vapour, or gas.

12.2 DIMENSIONING OF THIN-WALLED CYLINDRICAL TUBES AND VESSELS SUBJECTED TO INTERNAL PRESSURE.

The object of inspection is a thin-walled cylindrical vessel with sealed ends and the following properties:



D = 2r: inner diameter [mm]

v: wall thickness [mm]

p: internal pressure [N/mm2]

The weight of the medium the vessel, which can be very significant in some cases, is not considered in the calculations, we focus on the effect of the internal pressure p.

If the internal pressure p is not too large and the wall is thin, the distribution of the stresses in the wall can be assumed to be uniform.

The internal pressure induces a tangential σ_t stress in the vessel wall. Its value can be calculated by examining the equilibrium of an elementary unit cut from the wall of the vessel. Its length is 1 cm, and the height is described with the central angle, $\Delta\varphi$.

The pressure and the stress are in equilibrium like so:

$$\frac{pr\Delta\varphi}{2} = \sigma_t \cdot v \cdot \sin\frac{\Delta\varphi}{2}$$

Since $\Delta\varphi$ is small, $\sin\frac{\Delta\varphi}{2} \approx \frac{\Delta\varphi}{2}$, thus

$$\boldsymbol{\sigma_t = \frac{r \cdot p}{v} = \frac{Dp}{2v}}$$

We obtained **THE BOILER FORMULA.**

If the vessel is closed at the ends, there is not only a tangential but also an axial (axial) σ_a stress.

The resultant of the internal pressure on the end faces of a cylindrical vessel:

$$F_a = p\,\frac{D^2\pi}{4}$$

This force is in the longitudinal direction and can be assumed to be uniformly distributed over the cylinder surface $A_a = D \cdot \pi \cdot v$. Thus:

$$\sigma_a = \frac{F_a}{A_a} = p\,\frac{D^2\pi}{4D\pi v} = \frac{Dp}{4v}$$

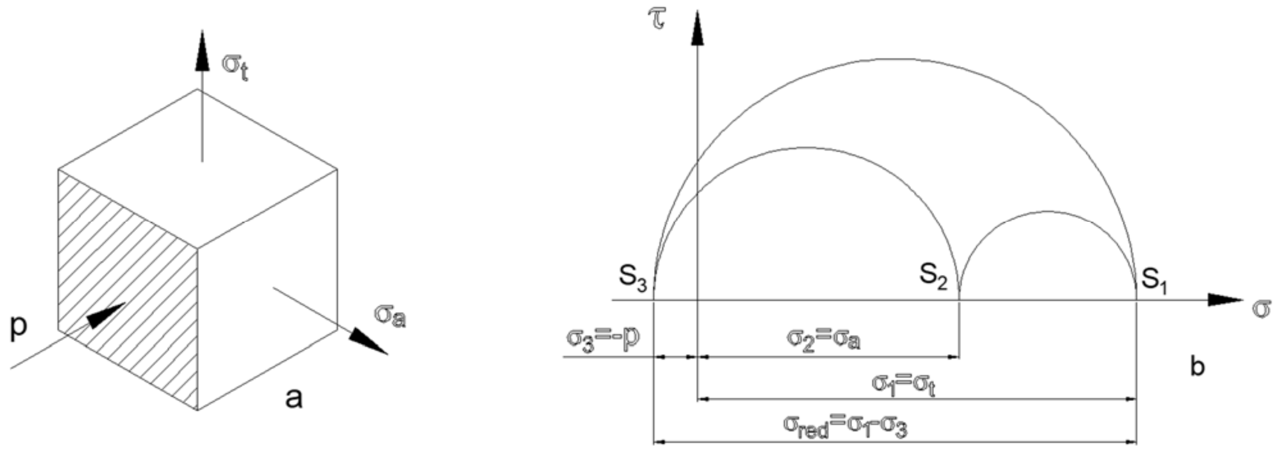
The axial σ_a stress is therefore just half of the tangential σ_t stress:

$$\sigma_a = \frac{\sigma_t}{2}$$

This is why pressurized tank failure happens on its side and not at its ends.

A vulgar example for pressurized tank failure on its side

The stresses can be visualized on an elementary unit taken from the vessel wall. The edges of the small cube are in the axial, radial and tangential directions.



Note, the since there is no shear stress, the stresses are principal stresses.

The sequence of principal stresses is:

$$\sigma_1=\sigma_t;\ \sigma_2=\sigma_a \text{ and } \sigma_3=-p.$$

The stress state of the elementary unit is spatial, thus, according to Mohr's theory, the reduced stress is expressed in terms of the principal stresses:

$$\sigma_{red}= \sigma_1 - \sigma_3.$$

The reduced stress is equal to the diameter of the largest Mohr circle $\sigma_1 = \sigma_t$ and $\sigma_3 = -p$ substituted:

$$\sigma_{red} = \sigma_t - (-p) = \frac{Dp}{2v} + p.$$

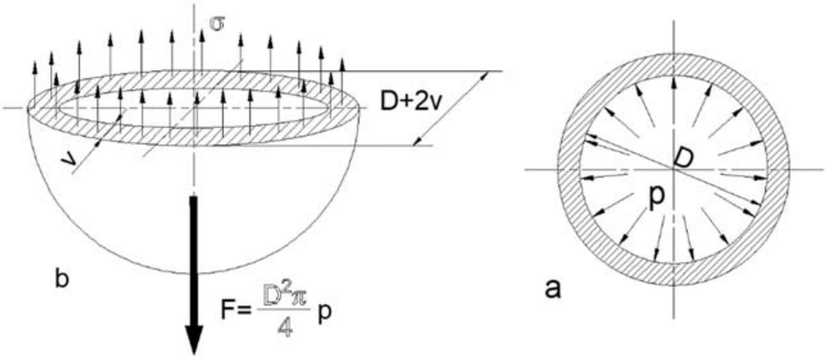
Since p is significantly smaller compared to σ_t, the sizing of vessels and pipes is in practice based on the boiler formula:

$$\sigma_t = \frac{Dp}{2v} \leq \sigma_{max}$$

$$\sigma_{max} = \sigma_t = \frac{Dp}{2v} = \sigma_{meg}$$

12.3 THIN-WALLED SPHERE SUBJECTED TO INTERNAL PRESSURE

A thin-walled sphere is subjected to an internal pressure p. If the sphere is cut into two parts along a diametral plane, the resultant force on the section is given by the internal pressure:

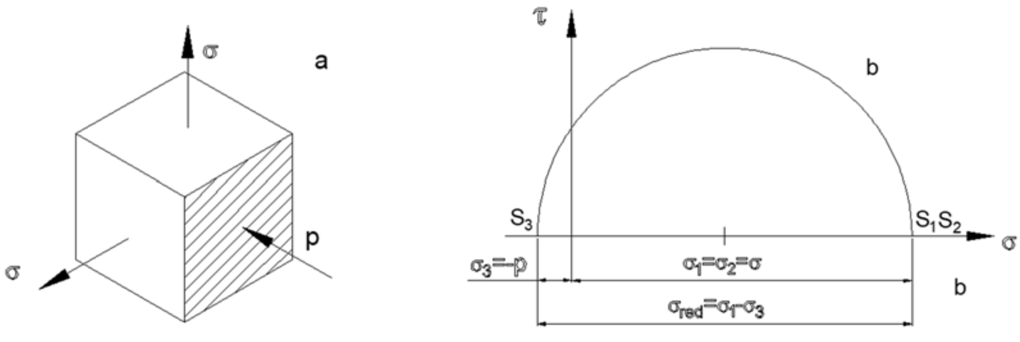


$$F = \frac{D^2\pi}{4} \cdot p$$

The tensile stress on surface A=D π v:

$$\sigma = \frac{F}{A} = \frac{D^2\pi p}{4D\pi v} = \frac{Dp}{4v}$$

The stresses can be visualized on an elementary unit taken from the vessel wall. The stresses are principal stresses.



Based on the Mohr circle, the reduced stress

$$\sigma_{red} = \sigma_1 - \sigma_3 = \frac{Dp}{4v} - (-p) = \frac{Dp}{4v} + p$$

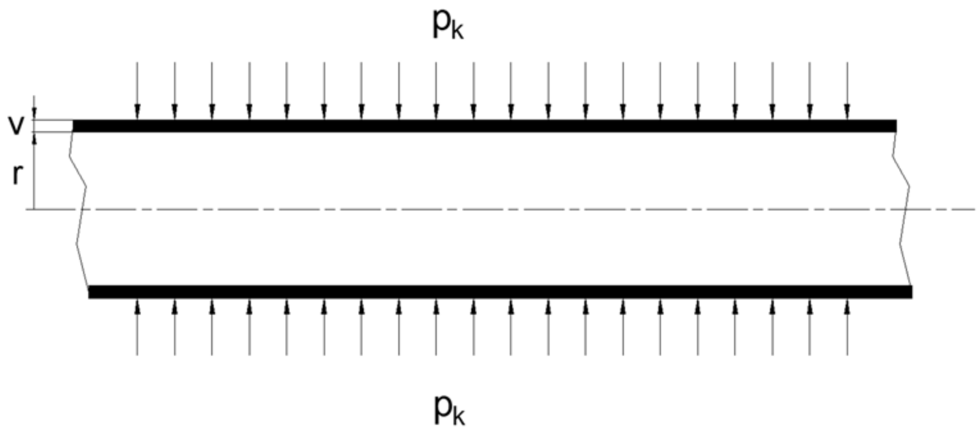
The value of p can be neglected relative to $\frac{Dp}{4v}$. Thus, the equation will be:

$$\sigma_{red} = \sigma = \frac{Dp}{4v} = \sigma_{meg}$$

The LNG carrier Aristidis I. transporting liquified natural gas in spherical containers.

12.4 PIPES SUBJECTED TO EXTERNAL PRESSURE

Pipes and vessels behave differently under external pressure than under internal pressure. While the internal pressure acting on the pipe or vessel tends to smooth out the irregularities, the external pressure can cause buckling. External pressure should preferably be applied only to cylindrical pipes.



For an infinitely long pipe, p_k is the critical external pressure at which the pipe will buckle.

$$p_K = \frac{E}{4}\frac{v^3}{r^3}$$

E is the elasticity coefficient of the pipe material,

v the wall thickness,

r is the inner radius.

The glass tunnel of the Tropicarium in Budapest is a good example for an externally loaded pipe

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